

Practice Exam 2 Solutions

This practice exam is similar in length, content, and format to the actual exam. This is not to say that the problems given here represent *all* of the concepts you will encounter on the actual exam, since it's difficult to "cover" all possible subjects in such a short exam! However, if you feel confident on your performance on this practice exam and you've gone over all homeworks and quizzes, you should feel confident about your upcoming performance on the actual exam.

In order to save paper, I have not included space for you to work out your solutions. (The actual exam will provide such space.) Rather, please complete solutions to the below problems on your own paper. The practice exam is worth a total of 100 points; the point value of each question is provided with that question.

1. (28 points total; 4 points each) Let $f(t) = \frac{t^2-t}{t^2+t-2}$.

- (a) Find all critical points of the function f .

Although you don't *need* to do this, a smooth first step is to factor both the top and the bottom and cancel common terms:

$$\frac{t^2-t}{t^2+t-2} = \frac{t(t-1)}{(t+2)(t-1)} = \frac{t}{t+2}$$

except when $t = 1$; at $t = 1$ we will have a hole.

Now we can find the derivative, using the Quotient Rule:

$$f'(t) = \frac{1(t+2) - 1(t)}{(t+2)^2} = \frac{t+2-t}{(t+2)^2} = \frac{2}{(t+2)^2}.$$

This is undefined when $t = -2$, and *never* equals 0 (since the numerator is a constant). Thus our critical point is $t = -2$. (Technically $t = 1$ could be counted as a critical point, too, but since the hole won't affect the increasing/decreasing nature of the graph, we don't need to count it.)

- (b) Find the intervals on which f is increasing, and the intervals on which f is decreasing.

We only have a single critical point, but we also have to keep in mind the point $t = 1$ because the function isn't defined there. Thus we need to consider the three intervals $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$.

Notice that the denominator of $f'(t)$ is always positive, since it's a square. The numerator is a positive constant, so the whole darned thing is always positive, and f is always increasing, on all three intervals given above!

- (c) Find all maxima and minima for the function f .

Since the function is always increasing, there are no maxima or minima, local or global!

- (d) Find the intervals on which f is concave up, and the intervals on which f is concave down.

Applying the Quotient Rule again and simplifying as much as possible, we obtain

$$f''(t) = \frac{-4}{(t+2)^3}.$$

As with the first derivative, this function is undefined at $t = -2$, and we keep in mind that the original function is undefined at $t = 1$ as well. Thus we have to consider the same intervals as in (a). Now, however, though the numerator is always negative, the denominator is negative precisely when $t < -2$. That is, $f''(t)$ is positive on $(-\infty, -2)$ and negative on $(-2, 1) \cup (1, \infty)$. Therefore f itself is concave upward on $(-\infty, -2)$ and concave downward on $(-2, 1) \cup (1, \infty)$.

- (e) Find all inflection points for the function f .

Strictly speaking there are no inflection points, since the only t -value at which the concavity changes is a point where the function f is undefined ($t = -2$).

- (f) Find all asymptotes for the function f .

As we already saw, there's a hole in the graph at the point $t = 1$, but because we can't cancel the term $t + 2$ from the denominator of the original function, there'll be a vertical asymptote at $t = -2$.

Furthermore, dividing both the top and the bottom by the highest power appearing in the denominator, we can analyze the behavior of f for large (positive and negative) values of t :

$$\frac{t^2 - t}{t^2 + t - 2} = \frac{\frac{t^2}{t^2} - \frac{t}{t^2}}{\frac{t^2}{t^2} + \frac{t}{t^2} - \frac{2}{t^2}} = \frac{1 - \frac{1}{t}}{1 + \frac{1}{t} - \frac{2}{t^2}}.$$

As t grows very large (*i.e.*, for $t \rightarrow \pm\infty$), the $\frac{1}{t}$ and the $\frac{2}{t^2}$ go to 0 in this last formula, and we obtain $\frac{1}{1} = 1$ in the limit. Thus $y = 1$ is a horizontal asymptote.

- (g) Sketch a graph of the function f using the information gathered in the previous six steps.

Put it together! Your graph should be above the horizontal asymptote $y = 1$ and rise to ∞ on the left side of the vertical asymptote $t = -2$, while on the other side of $t = -2$ it should rise from $-\infty$ and approach the horizontal asymptote $y = 1$ from below.

2. (12 points total; 6 points each)

- (a) Name all of the types of discontinuity a function can have.

A function can have

- i. *infinite* discontinuities, *i.e.*, vertical asymptotes,
- ii. *removable* discontinuities, at which defining or redefining the function at a single point fixes the problem (filling in a hole),
- iii. *jump* discontinuities, places where the function jumps from one value to another and removing the discontinuity is impossible, and
- iv. *oscillatory* discontinuities, as in $\sin(1/x)$, whose graph dives up and down and up and down and up and down infinitely often in a finite space.

- (b) Draw the graph of a *single* function which exhibits each of the types of discontinuity you mentioned in part (a).

The descriptions above should remind you of the problems that arise. Give it a go!

3. (10 points) Farmer Bob needs to build a fence to enclose his guinea pig pasture. The pasture is bordered on one side by a river, so he only needs to put fencing on three sides of the pasture, which must be rectangular. If Farmer Bob has 100 meters of fencing, what's the *largest* area he can enclose? (*Hint*: draw a diagram and set up some notation. Then figure out how to compute the area and perimeter of the enclosure...to maximize the area you must then...)

Let's agree to let the *length*, ℓ , of the enclosure be the length of the side parallel to the river, and the *width*, w , to be the length of the sides perpendicular to the river. Then we must maximize the area $A = w\ell$ knowing that the perimeter $P = 2w + \ell$ must be 100.

We may solve $100 = 2w + \ell$ for ℓ , obtaining $\ell = 100 - 2w$. Replacing the ℓ in $A = w\ell$ with this value, we get

$$A = w\ell = w(100 - 2w) = 100w - 2w^2.$$

Notice that the area is undefined for $w < 0$ and w cannot be more than 50, so we consider the interval $[0, 50]$.

To maximize $A(w)$ on this interval, we consider value of A at the critical points. Since $A'(w) = 100 - 4w = 0 \Rightarrow 100 = 4w \Rightarrow w = 25$, we have three critical points, $w = 25$, $w = 0$, and $w = 50$. (These last two are critical points because they're endpoints of the interval.)

It's easy to see that $A(0) = A(50) = 0$, and the derivative $A'(w)$ is positive on $(0, 25)$ and negative on $(25, 50)$. Thus $w = 25$ is a local (and global!) maximum, giving the value $A(25) = 1250 \text{ m}^2$.

4. (30 points total; 6 points each) Compute the derivative of each of the following functions (you *may* use shortcut formulas!):

(a) $\frac{\sin(t)}{t}$

Use the Quotient Rule:

$$\left(\frac{\sin(t)}{t}\right)' = \frac{\cos(t) \cdot t - \sin(t) \cdot 1}{t^2} = \frac{t \cos(t) - \sin(t)}{t^2}.$$

(b) $\frac{t}{\sin(t)}$

Quotient Rule again!:

$$\left(\frac{t}{\sin(t)}\right)' = \frac{1 \cdot \sin(t) - t \cdot \cos(t)}{\sin^2(t)} = \frac{\sin(t) - t \cos(t)}{\sin^2(t)}.$$

(c) $e^x \cos(x)$

Now it's the Product Rule:

$$(e^x \cos(x))' = e^x \cdot \cos(x) + e^x \cdot (-\sin(x)) = e^x(\cos(x) - \sin(x)).$$

(d) $\frac{xe^x}{x^2+1}$

Now we've got to combine them, using the Product Rule *inside of* the Quotient Rule:

$$\left(\frac{xe^x}{x^2+1}\right)' = \frac{(1 \cdot e^x + x \cdot e^x)(x^2+1) - xe^x \cdot 2x}{(x^2+1)^2} = \frac{e^x(x^3 - x^2 + x + 1)}{(x^2+1)^2}$$

after simplifying. (Any equivalent expression would be acceptable.)

(e) $\cot(x)$ (*Hint*: $\cot(x) = \frac{\cos(x)}{\sin(x)}$)

Quotient Rule one more time:

$$(\cot(x))' = \frac{-\sin(x) \cdot \sin(x) - \cos(x) \cdot \cos(x)}{\sin^2(x)} = \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)},$$

where in the last step we've used the most useful of all trig identities, $\sin^2(\theta) + \cos^2(\theta) = 1$ for any θ . The last formula above is usually written $-\csc^2(x)$, our final answer.

5. (10 points) As budget analyst for Crystal Curios, you find that the cost (per unit) of manufacturing u thousand glass unicorns is given by $C(u) = u^3 - 5u^2 + 7u$ dollars. Find the number of unicorns your company must manufacture if it's to minimize the cost of production, per unicorn produced. (Be sure to *prove* that you've found a minimum!)

Since we can't manufacture a negative number of unicorns, we need only consider the interval $[0, \infty)$. We differentiate, obtaining $C'(u) = 3u^2 - 10u + 7$. We can apply the quadratic formula to find the roots:

$$u = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(7)}}{2(3)} = \frac{10 \pm \sqrt{16}}{6} = \frac{10 \pm 4}{6} = 1, \frac{7}{3}.$$

We thus consider the intervals $(0, 1)$, $(1, \frac{7}{3})$, and $(\frac{7}{3}, \infty)$. The derivative $C'(u)$ is positive on the first and last of these intervals and negative on the middle one, meaning that the value $u = 1$ is a local maximum and the value $u = \frac{7}{3}$ is a local minimum.

There's a little bit of interpretation one must do here, since technically it's cheaper to not manufacture unicorns at all; it's not until one manufactures about 333 (that is, 0.333 thousand!) unicorns that the cost per unit is greater than the local minimum. However, if you're making more than 333 unicorns, you're best off making about 2333 of them.

6. (10 points) Explain in your own words what the following expression means:

$$\lim_{x \rightarrow 2} f(x) = 4.$$

The above expression says that as x nears the value 2, the function $f(x)$ (whatever it is) approaches the value 4. Put another way, we can make $f(x)$ as close as we would like to to the value 4 by considering values of x close enough to 2.