MATH 191, Section 2
Calculus I
Spring 2009

Practice Exam 1 Solutions

This practice exam is similar in length, content, and format to the actual exam. This is not to say that the problems given here represent *all* of the concepts you will encounter on the actual exam, since it's difficult to "cover" all possible subjects in such a short exam! However, if you feel confident on your performance on this practice exam and you've gone over all homeworks and quizzes, you should feel confident about your upcoming performance on the actual exam.

In order to save paper, I have not included space for you to work out your solutions. (The actual exam will provide such space.) Rather, please complete solutions to the below problems on your own paper.

The practice exam is worth a total of 100 points; the point value of each question is provided with that question.

(1) (16 points) The paragraph should be completed as follows:

The derivative of the function $f(x)$, denoted $\frac{df}{dx}$ (or $f'(x)$), gives the instantaneous rate of change of the function f with respect to x. We can also interpret the derivative at a point as the slope of the tangent line to the function's graph at that point. Formally, computing derivatives requires that we find the limit of the quantity $\frac{f(x+h)-f(x)}{h}$ as h goes to 0, but so far we've just been simplifying this formula and plugging in $h = 0$ when safe to do so.

(2) (12 points) Using the *definition* of the derivative (and ***no*** shortcuts!), compute the derivative of the function $f(t) = \frac{2}{t}$.

We're asked to use the limit definition of the derivative. Let's go ahead and do just that:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{2}{t}\right) &= \lim_{h\to 0} \frac{\frac{2}{t+h} - \frac{2}{t}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2t}{t(t+h)} - \frac{2(t+h)}{t(t+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2t-2t-2h}{t(t+h)}}{h} \\
&= \lim_{h\to 0} \frac{-2h}{ht(t+h)} \\
&= \lim_{h\to 0} \frac{-2}{t(t+h)} \\
&= -\frac{2}{t^2}.
\end{aligned}$$

(3) (24 points total; 6 points each) Find the derivative of each of the following functions. (You *may* use shortcut formulas!)

Here are the deratives requested; these all follow from straightforward application of the shortcut rules we've established:

(a) $2\cos(x) + 3\sin(x)$
(b) $4e^x - \frac{1}{2}x^{-1/2} = 4e^x - \frac{1}{2\sqrt{x}}$
(c) $2000x + 2000x^{999}$
(d) $25x^4 + 16x^3 - 9x^2 + 4x - 1$

(4) (10 points) Explain briefly (but carefully) how you can use the derivative of a function f in order to determine the intervals on which f is increasing.

Since the derivative of a function f represents the slope of the tangent line to the graph of the function at a particular point, we can use the derivative to tell whether the function's slope is rising or falling. If the derivative is positive, then the slope of the function f is positive, meaning the function is increasing. Therefore, in order to find the intervals on which the function f is increasing, we merely need to find the intervals on which the derivative is positive. (As you may recall from precalculus (and from one or two homework problems this semester), the best way to do this is to first find the points at which the derivative is 0 and then test each of the intervals determined by these points to see whether the derivative is positive or negative on the respective interval.)

(5) (20 points total; 10 points each) Let $f(x) = e^x - x$.

(a) Find the equation of the tangent line to the graph of f at the point $x = 1$.

To find the tangent line at the point $x = 1$, we use the point-slope equation for the line: $y - y_1 = m(x - x_1)$.
Finding m is as easy as finding the derivative. Since $f'(x) = e^x - 1$, we see that $f'(1) = e^1 - 1 = e - 1$. (This doesn't simplify any further.)
To find our point (x_1, y_1), we only need to plug $x_1 = 1$ into our formula for f:

$$y_1 = f(x_1) = f(1) = e^1 - 1 = e - 1$$

also.
Putting this together, we get

$$y - (e - 1) = (e - 1)(x - 1) \Rightarrow y = (e - 1)(x - 1) + e - 1 \Rightarrow y = (e - 1)x.$$

Any equation equivalent to this one would be a valid answer.

(b) At what values of x is the tangent line to the graph of f *horizontal*?
To determine where the tangent line to f's graph is horizontal, we must find out where the derivative $f'(x) = e^x - 1$ is equal to 0. But

$$f'(x) = 0 \Leftrightarrow e^x - 1 = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0.$$

Thus we obtain a horiztonal tangent line when $x = 0$.

(6) (10 points) Find the domain of the function $g(x) = \frac{\sqrt{x+2}}{x^2+2x-3}$.

There are two rules for finding domains at work here. First, we note that we cannot take the square root of a negative number. Thus in order for the numerator to be defined, we must have $x + 2 \geq 0$, or $x \geq -2$. Second, in order that we do not divide by 0, we must be sure that the denominator $x^2 + 2x - 3$ is not 0. To find the denominator's zeroes, we factor:

$$x^2 + 2x - 3 = (x + 3)(x - 1).$$

Thus the denominator is 0 when $x = -3$ or $x = 1$. We can ignore the first zero since it's less than -2 anyway, but we need to be sure to exclude the second one.

Putting it all together, our domain is $[-2, 1) \cup (1, \infty)$, beginning at $x = -2$ and skipping only $x = 1$.

(7) (8 points) Suppose we are asked to come up with a function $C(r)$ which gives an estimate for the number of lunar craters with radius r. Which of the following functions is the better choice?

$$C(r) = r^2 \quad \text{or} \quad C(r) = \frac{1}{r^2}$$

Please explain your answer briefly. (*Hint:* think about what the graphs of these functions look like, especially for large value of r.)

No need for a planetary physicist: a little bit of physical intuition is all we need here. All we need to realize is that one of these functions, r^2, *increases* as r gets larger, while the other function, $\frac{1}{r^2}$, *decreases*. Since it's reasonable to expect that the number of craters with a given radius goes down as the radius we're measuring gets larger, we probably want the second of these functions, so $C(r) = \frac{1}{r^2}$ is the better answer.

Incidentally, it turns out that there is a sort of "power law" for crater density: actual models for crater counts suggest a function like $C(r) \approx \frac{k}{r^a}$ for some constant k and some number a around 3 or 4.