MATH 191, SECTION 2
CALCULUS I
SPRING 2009

A Non-Strang Interlude: Exponential Functions

Okay, now onto another nontrivial sort of function: ______________ functions. Much as we did with $f(x) = \sin(x)$ we can use the definition of the derivative as a limit to compute $f'(x)$ when $f(x) = a^x$ for some fixed constant $a > 0$.

Let's get the ball rolling and just follow our noses, using a familiar rule for dealing with exponents:

So it all depends upon what the limit $\lim_{h \to 0} \frac{a^h - 1}{h}$ is.

Examples. Let's use *Mathematica* to experiment and estimate what this last limit might be.

(1) Use *Mathematica* to plot the graph of $\frac{2^h - 1}{h}$ and estimate the value of this expression when h gets close to 0. (*Hint*: you can get more and more accurate estimates by adjusting the domain over which you plot your function. Try to get accuracy to 2 places after the decimal.)

You can sketch a graph of the function $\frac{2^h-1}{h}$ and give your estimate for $\lim_{h \to 0} \frac{2^h - 1}{h}$ in the space below:

What does your limit say about $\frac{d}{dx}(2^x)$?

(2) What about when $a = 3$? Follow the process above and give your results below:

(3) What about when $a = 2.5$?

Our investigation above suggests that there should be some number a in between 2.5 and 3 for which the limit $\lim_{h \to 0} \frac{a^h - 1}{h}$ is 1. In this case $f(x) = a^x$ gives $f'(x) = a^x$, the original function. Wouldn't that be cool?!!?

Well, there is such a number. It's our good old friend, e, the base of the natural logarithm. In fact, e is simply *defined* to be this number, and $e \approx 2.71828182...$. That is, we have the

Definition. Denote by e the unique number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Thus, if $f(x) = e^x$ then $f'(x) = e^x$ as well. The constant multiples of this function are the *only* functions equal to their own derivatives!

Examples. Use your newfound intellect to complete the derivatives of the following functions.

(1) $f(x) = e^x + 3x^2$

(2) $h(r) = \frac{2}{r} + 4e^r$

(3) $g(x) = \sqrt{x} - e^{x+1}$ (*Hint:* Note that $e^{x+1} = e \cdot e^x$, and e is just a constant!)

Homework. The following homework is due at 5:00 p.m. on *Friday, February 20th.*

(1) Find the derivatives of each of the following functions.
   (a) $\sin(x) + e^x$
   (b) $x^7 - e^{x+2}$
   (c) ex^2 (*Hint:* e is simply a constant...)
   (d) $\frac{3}{x^5} - \pi e^x$

(2) Find a function whose derivative is $x^6 + \cos(x) - 3e^x$.

(3) Find an equation for the tangent line to the graph of the function $f(x) = 2e^x$ at the point $x = 1$.

(4) The function $f(x) = e^x - x$ has a single *global minimum*; that is, there is a single point at which the function f obtains it lowest possible value. At this point x, the graph of f must "bottom out" and have a horizontal tangent line. Find the x-coordinate of the minimum using the derivative of f.

(5) **(Extra credit)** The *Mathematica* notebook "Mysterious.nb" (downloadable from the class website) contains the commands that will graph the values of the limit $\lim_{h \to 0} \frac{a^h - 1}{h}$ for various values of a. Use this notebook to come up with a conjecture for the *familiar function* F such that $F(a) = \lim_{h \to 0} \frac{a^h - 1}{h}$. (*Hint:* we know that $F(e) = 1$. What familiar function has this property?)