MATH 191, SECTIONS 1 AND 3
CALCULUS I
FALL 2009

Section 4.3: The Mean Value Theorem

Today's focus is probably the second (maybe third) most important theorem from all of calculus. (Only the Intermediate Value Theorem, which we mentioned when we defined continuous functions, and the Fundamental Theorem of Calculus, which we'll see briefly at the end this semester, might outshine it in importance.) We're going to sneak up on the Mean Value Theorem, starting with a miniature version of it:

Rolle's Theorem. Suppose that the function f

1. is ______________ on the closed interval $[a, b]$,
2. is ______________ on the open interval (a, b), and
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) =$ ___ .

Go ahead and use the space provided for you below to sketch a

Proof of Rolle's Theorem. (*Hint*: it might not hurt to include a graph or two to help visualize things, as well!)

Note. If $f(t)$ represents a position function, an interesting corollary of Rolle's Theorem is that the velocity of the moving object must be 0 at some point, provided the object begins and ends at the same point.

Note what Rolle's Theorem is saying about tangent and secant lines: there is some point c inside the interval (a, b) at which $f'(c) = 0$. That is, the ____________ line at this point is parallel to the ____________ line through $(a, f(a))$ and $(b, f(b))$. More generally, if f is sufficiently well-behaved, we can find tangent lines which run parallel to more general secant lines. This is purport of the Mean Value Theorem:

Mean Value Theorem. Suppose that f is ____________ on the interval $[a, b]$ and differentiable on the interval ________ . Then there is some number c on (a, b) such that ________ $= \frac{f(b)-f(a)}{b-a}$.

Before proving this theorem, take a moment to understand what it says: if f is "nice," then there's a point c in the interval (a, b) at which the tangent line runs parallel to the secant through $(a, f(a))$ and $(b, f(b))$, regardless of this last line's slope. Interpreting further, there's some point c at which the *instantaneous* rate of change of f with respect to x is equal to the *average* rate of change over the entire interval.

Notice that we aren't guaranteed that we can actually *find* the value of c that makes the MVT work; we're just told that it exists somewhere.

Proof of the Mean Value Theorem. Instead of considering the function f itself, let's replace f with the new function $F(x) = f(x) - f(a) - \left(\frac{f(b)-f(a)}{b-a}\right)(x - a)$.

Note. This function might look mysterious, but it's really obtained by taking the difference between the function f itself and the line drawn between the two points $(a, f(a))$ and $(x, f(x))$!

1. Is F differentiable? Where? Is it continuous? Where?

2. What's $F(a)$? $F(b)$?

3. What does Rolle's Theorem now say about F? Can you translate this into a statement about f?

Here are some nice applications of the MVT:

Theorem. If $f'(x) = 0$ for all x on (a, b), then f is constant on the interval (a, b).

Proof. This isn't too bad: all we need to do is show that if $x_1 < x_2$ are two points in (a, b), then $f(x_1) = f(x_2)$. Here's some room for you to check that MVT is just what we need:

In exactly the same way you can prove

Theorem. Suppose that f is differentiable on the open interval (a, b). Then

1. if $f'(x) > 0$ for all x in (a, b), then f is ____________ on (a, b), and
2. if $f'(x) < 0$ for all x in (a, b), then f is ____________ on (a, b).

This observation gives us a means of further interpreting what's going on at any critical point c. If $f'(c) = 0$ and the derivative goes from positive to negative at $x = c$, then we increase up to c and we decrease going away, so we must have a local ____________ at $x = c$. We can summarize this and a similar fact in the following

Theorem (First Derivative Test). Let f be a differentiable function and let c be a critical point of f. Then

1. if f' changes from ____________ to ____________ at $x = c$, then c is a local maximum for f,
2. if f' changes from ____________ to ____________ at $x = c$, then c is a local minimum for f, and
3. if f' does not change sign at $x = c$, then c is not a local extremum.

Example. Analyze the behavior of the function $f(x) = x^{5/2} - x^2$.

Notice that we can even tell what's happening on the intervals defined by the critical points of f, summarizing it in a nice little diagram:

Essentially what we've done above is found the intervals of increase and decrease for the function f.

Let's try another

Example. Find the increasing and decreasing behavior of the function $g(t) = t - \ln(t)$, for t in $(0, 5)$.

Homework from Section 4.3 (pp. 236-238): numbers 12, 13, 20, 27, 33, 36, 43, 52, 60, and 64. This homework is due on *Friday, November 13th.*