Section 4.2: Extreme values

We've already made the following

Observation. Any function $y = f(x)$ seems to obtain its maximum and minimum values at values of x for which $f'(x) =$ _____ .

Although this observation is not always true, it is true often enough to form the basis of a technique for finding the ______________ *values* of a function. Before we describe our technique, we need a few

Definitions. Given a function f defined on an interval I, we say that

1. $f(a)$ is an ______________ (or *global*) ______________ for f on I if for all x in I, $f(x) \leq f(a)$.
2. $f(a)$ is an ______________ (or *global*) ______________ for f on I if for all x in I, $f(x) \geq f(a)$.

Although it may seem like these values (called ______________ *values* collectively) always exist, this may not be the case if f is not continuous:

Example. Draw the graph of a function f on an interval I which fails to have (1) an absolute maximum because f is not continuous at a certain point, and (2) an absolute minimum because I is an open interval.

Nevertheless, if we avoid the two issues appearing in the previous example, all will work out well:

Extreme Value Theorem. If f is continuous on a closed interval $I = [a, b]$, it will always obtain its absolute maxima and minima.

Sometimes we need not find the *absolute* extrema, but merely the values of x at which f takes on the greatest or least values "nearby."

Definition. Let f be defined on the interval I.

1. We say that f has a *local* (or ____________) *maximum* at a if $f(x) \leq f(a)$ for all x in the domain of f on some open interval around a.
2. We say that f has a *local* (or ____________) ____________ at a if $f(x) \geq$ ____ for all x in the domain of f on some open interval around a.

Example. In the space below, draw the graph of a function f that has (1) a global maximum at $x = -1$, (2) no global minimum, (3) local maxima (that are *not* global maxima) at $x = 0$ and $x = 5$, and (4) local minima at $x = -\frac{1}{2}$ and $x = 3$.

What's going on at every one of the extrema you drew above? Most likely $f'(c) = 0$ for every such extremum c...if you're creative, maybe you drew something called a *cusp*, a point at which the function comes to a sharp thorn-shaped point; $f'(c)$ does not exist at such points. In either case c was an example of the following

Definition. The value $x = c$ is a ____________ *point* for the function f if $f'(c) = 0$ or $f'(c)$ does not exist.

These points are called ______________ points for a reason:

Fermat's Theorem. If f has a local extremum at the point $x = c$, then c is a critical point of f.

We won't prove Fermat's Theorem, but a proof is given on page 222 of your textbook (it's not too hard to follow.)

What's Fermat's Theorem mean for us? First of all, it makes the observation with which we began this section more precise. Second, it gives us a sweet method of finding the extreme values of a function on an interval.

Finding extrema. Let f be a function defined on a closed interval $I = [a, b]$. To find the extreme values of f on I,

1. Find all critical points c by computing f' and determining where it (a) equals 0 or (b) does not exist.
2. Compute $f(c)$ for each of the values from (1).
3. Compute $f(a)$ and $f(b)$.
4. Compare! The biggest value(s) you found will give you maxima; the smallest will give you minima.

Examples.

1. Find the extrema of the function $g(t) = 2t^3 - 3t^2 - 12t + 1$ on the interval $[-3, 3]$.

2. Find the extrema of the function $f(x) = x^2 e^x$ on the interval $[-3, 1]$.

3. Suppose that the concentration of methylphenidate in a patient's bloodstream (measured in mg/cm^3) is given by $C(t) = \frac{0.016t}{t^2+4t+4}$, when t is measured in hours. At what time is the concentration the highest?

We will next use the derivative f' to begin understanding the structure of f on intervals, not merely at single isolated points.

Homework from Section 4.2 (pp. 226-229): numbers 1, 4, 6, 11, 12, 23, 28, 33, 38, 43, 48, 53, and 73. This homework is due on *Friday, November 6th.*