MATH 191, SECTIONS 1 AND 3
CALCULUS I
FALL 2009

Section 4.1: Linearization

Sometimes, when given a function $y = f(x)$ we may be interested in examining the effect of a small change in x on y. That is, if we "perturb" x by a small amount, what effect does this perturbation have on the value of y? If our function is a complicated one, computing the change in y *precisely* may be difficult.

Example. Suppose the cost $C(x)$ of manufacturing gold-plated toothbrushes depends on the number, x, of toothbrushes we manufacture, and is given by $C(x) = 0.0001x^3 + 0.02x - .05$. What's the difference in cost between manufacturing 1000 toothbrushes and 1001 of them?

Here the numbers we had to work with were unwieldy; it would be nice to work with a simpler function that gives a decent approximation to the change we seek.

To obtain such an approximation, at, say, the point $x = a$. we look at the definition of the derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\Delta f}{x - a}.$$

Thus if $x - a$ is very small, we can estimate:

$$\Delta f \approx f'(a) \cdot \underline{\hspace{4cm}} \; .$$

Writing out $\Delta f = f(x) - f(a)$, we can even approximate the value of $f(x)$:

$$f(x) \approx \underline{\hspace{6cm}} \; .$$

Example, redux. Returning to our previous example, *estimate* the same difference you computed before by letting $a =$ _________ and $x =$ _________ .

The estimate for $f(x)$ we obtained in the last formula on the previous page is called the ______________ *approximation* to $f(x)$, or the _______________________ of $f(x)$, since it's obviously a ______________ function. It's sometimes denoted (as in your textbook) $L(x)$, or $L_f(x)$.

These approximations are particularly useful for estimating roots and radicals:

Example. Use the linearization of the function $f(x) = \sqrt[3]{x}$ to estimate the value of $\sqrt[3]{1001}$. (*Hint:* let $a =$ _________ and $x =$ _________ .)

Example. The *resistance* R of a wire (that is, the tendency of its material to oppose the flow of electrical charge) increases as the temperature T of the wire increases. Suppose we know that for a length of copper wire R has the value 15 ohms (Ω) at $T = 20°\text{C}$, and that $\frac{dR}{dT}|_{T=20}$ is $0.06\ \Omega/°\text{C}$. Estimate the resistance in a copper wire at each temperature $T = 21$, $t = 22$, and $T = 23$.

If we'd like to know how far off our estimates are, assuming we know how to compute the actual value we're trying to approximate, we can compute the percentage error in our estimate with a very simple formula:

$$\text{percentage error } = \frac{\text{error}}{\text{actual value}}.$$

Example. Compute the percentage error in the approximation to $\sqrt[3]{1001}$ we obtained above. (*Hint:* first find the *actual* value using a calculator...)

Later we will be able to determine upper bounds (that is, "worst cases") for percentage error in given approximations.

Homework from Section 4.1 (pp. 217-219): numbers 5, 10, 13, 23, 26, 34, 43, and 48. This homework is due on *Friday, November 6th.*