Section 3.9 and 3.10: Derivatives of Inverses

It turns out that if we know the derivative of a function f, it's not all that hard to find the derivative of its inverse, provided the inverse exists:

Theorem. Let f be a differentiable one-to-one function with inverse $g(x) = f^{-1}(x)$. Then if for any b in g's domain, if $f'(g(b)) \neq 0$, then $g'(b)$ exists and $g'(b) = \frac{1}{f'(g(b))}$.

That is, to find the derivative of the inverse at a point b, we take the reciprocal of the derivative of the *original* function at the point $g(b)$.

Why on *Earth* should this be true?!?

Think about it in terms of the graphs: the graph of the inverse function g is obtained from the graph of f by flipping it around the line ______________ , as in the following picture:

Note what happens to the tangent line to $g(x)$ at the point b as we flip it back to the graph of f: it falls on top of the tangent line to $f(x)$ at the point $g(b)$.

Since in forming the inverse we're exchanging x and y, these two tangent lines have slopes that are ______________ of one another. This leads directly to our formula above.

Examples.

1. Find the derivative $g'(1)$ if g is the inverse of the function $f(x) = x + e^x$. (*Note:* $g(1) = 0$. Why is this?)

2. Find the derivative of the inverse sine function, $g(x) = \sin^{-1}(x)$, simplifying the formula as far as you can.

3. Find the derivative of the inverse cosine function, $g(x) = \cos^{-1}(x)$, simplifying the formula as far as you can.

4. Use the formula you obtained above to find the equation of the tangent line to the graph of $\cos^{-1}(x)$ at the value $x = \frac{1}{2}$.

We can also use our "inverse derivative" methods to find the derivative of the natural log, knowing that it's the inverse to the exponential function $f(x) = e^x$. Compute $\frac{d}{dx}(\ln(x))$ in the space below:

We can also work with more general exponents and logs! In the space below, use the fact that $b = e^{\ln(b)}$ for any $b > 0$ to find the derivative $\frac{d}{dx}(b^x)$:

Remember that we showed at one time that $\frac{d}{dx}(b^x) = \lim_{h\to 0} \frac{b^h-1}{h} \cdot b^x$...now the formula you've just found allows you to compute that limit!:

$$\lim_{h\to 0} \frac{b^h - 1}{h} = \underline{\hspace{3cm}} \quad .$$

Pretty cool, huh? You should be able to use the newfound differentiation formulas, combined with our old stand-bys, to find derivatives of functions involving logs and inverse trig functions in products and quotients and compositions:

Examples. Find the derivative of each function given below.

1. $f(x) = x\ln(x)$

2. $g(t) = \frac{\sin^{-1}(t)}{t}$

3. $H(x) = \frac{e^x}{x\ln(x)}$ (*Hint:* use the derivative you computed in (1) in working with the bottom here...)

4. $g(s) = 2^{\sin(s)}$

5. $f(x) = \cos^{-1}(e^{\tan(x)})$

By the way, your textbook has formulas for the other inverse trig derivatives, by *far* the most important of which is

$$\frac{d}{dx}(\tan^{-1}(x)) = \underline{\hspace{3cm}}\,.$$

There's one more technique involving logarithms that often comes in handy for differentiating *really* crazy functions, like $y = x^x$. This technique is called ________________ *differentiation.* It's best understood by doing an

Example. Find the derivative of $f(x) = x^x$ by (1) writing $y = x^x$, (2) taking the natural log of both sides, and (3) differentiating the result implicitly.

Here's another example in which logarithmic differentiation simplifies matters considerably:

Example. Use logarithmic differentiation to compute $f'(x)$ if $f(x) = \frac{x(x+1)^3}{(3x-1)^2}$.

Homework from Section 3.9 (pp. 191-192): numbers 11, 15, 19, 24, 27, 31, and 33; and **from Section 3.10 (pp. 197-199):** numbers 3, 6, 9, 12, 15, 18, 29, 32, 37, and 44. This homework is due on *Friday, October 23rd.*