Section 3.8: Implicit differentiation

Most of the time so far we've been given a relationship between two variables, say x and y, by specifying *explicitly* how one depends on the other. For instance, we may write $y = \sin(x)$ to say *explicitly* that y is a function of x.

Sometimes data will be given to us in such a fashion that a relationship between two or more variables is *implicit* in a formula, without that formula specifying the way one variable depends on the other as a function. Sometimes we can even *graph* this relationship, even if the resulting graph is not that of a function.

Example. Suppose we know that $x^2 + y^2 = 1$. What is the graph of this relationship? Draw this graph in the space below. Why is it *not* the graph of a function?

Even though, as in the above example, we may not have an explicit description of y as a function of x, we may still like to know how fast y changes corresponding to a small change in x. There are (sometimes!) two ways to go about doing this.

1. **Solve for y in terms of x.** We could simply solve for y in terms of x and differentiate the resulting explicit formula (or formulas). Try this with the example given above, in which $x^2 + y^2 = 1$:

Notice that here we need two formulas, because undoing the square leads to two potential values for y.

2. **Differentiate ____________________ .** Instead, let's just *think* of y as a function of x, in which case anytime a function of y occurs in our formula, we must think of it as a function *of a function* of x. Such compositions require the ____________ Rule.

   Let's try this out with the example above:

   The beauty of this method is that we only needed to consider *one* case, even though an explicit solution required two! Our single solution captures all of the information contained in *both* of the other cases, as you should check.

Our method of ______________ *differentiation* to find $\frac{dy}{dx}$ will go as follows:

1. Begin with an implicit relationship involving your variables. (Often this will look like $f(x, y) = g(x, y)$ for some expressions f and g involving both variables.)
2. Differentiate both sides of the expression with respect to x, being careful to use the ____________ Rule to handle any function of y.
3. Solve for $\frac{dy}{dx}$.

Sometimes writing y' instead of $\frac{dy}{dx}$ makes your work easier to read.

Examples. Find $\frac{dy}{dx}$ implicitly in each of the examples below.

1. $x^4 + y^4 = 1$

2. $e^{xy} = x + y$

3. Find the points on the graph of $y^2 = x^3 - 3x + 1$ where the tangent is horizontal by setting $y' = 0$ and finding the x values that arise.

This last example is an instance of an ________________ *curve*; these relationships play a *major* role in modern *number theory*, the study of properties of the natural numbers. In particular, these curves were a crucial stepping stone in the proof of *Fermat's Last Theorem*, completed in 1994.

Homework from Section 3.8 (pp. 184-187): numbers 4, 9, 12, 15, 18, 21, 24, 30, 33, 35, 38, 45, and 59. This homework is due on *Friday, October 16th.*