MATH 191, Sections 1 and 3
Calculus I
Fall 2009

Section 3.7: Chain Rule

At this point we have formulas which allow us to find derivatives for a number of basic functions, as well as of functions that are formed by combining simpler functions in various ways (addition, multiplication, division, *etc.*). One means of combining functions we've not yet analyzed is *composition.* As composition always proves tricky for a number of people, it might not hurt to refresh ourselves with a few

Examples. Each of the following functions can be decomposed as a composition of two or more functions ($f \circ g$ or $f \circ g \circ h$ or some such nonsense). Write each of the functions in the decomposition explicitly.

1. $F(x) = \sin(x^2)$

2. $G(x) = \sin^2(x)$

3. $H(t) = \sqrt{e^t}$

4. $J(s) = e^{\sqrt{s^2+1}}$

We're now ready to introduce the _________ *Rule*, for computing derivatives of compositions:

_________ **Rule.** Let f and g be functions such that g is differentiable at x and f is differentiable at $g(x)$. Then $f \circ g$ is differentiable at x, and

$$(f \circ g)'(x) = \underline{\hspace{3cm}} \ .$$

Note. Just as when considering continuity of compositions, we don't care what happens with f at x, but only at $g(x)$. The most common mistake in applying the Chain Rule is to overlook this and use the **FALSE** formula $f'(x)g'(x) \neq (f \circ g)'(x)$.

Although we won't prove the Chain Rule carefully, it's worthwhile to think about why it *should* be true, and Leibniz's notation really helps us see this.

Let's write $u = g(x)$ and $y = f(y)$, so that $y = f(g(x)) = (f \circ g)(x)$. You should be able to translate the formula $(f \circ g)'(x) = f'(g(x))g'(x)$ above into Leibniz's notation for derivatives, using x, u, and y:

Wow! Notice how this notation suggests what the formula *should* be, merely by multiplying the differentials as though they're really fractions! Intuitively this makes sense, too. Suppose that $a = g'(x) = \frac{du}{dx}$ and $b = f'(u) = \frac{dy}{du}$. Then, approximately, a small change, Δx, in x results in a change of $\Delta u \approx a\Delta x$ in u, and a small change, Δu, in u results in a change of $\Delta y \approx b\Delta u$ in y. Putting this together,

$$\Delta y \approx b\Delta u \approx \underline{\hspace{4cm}}\ .$$

Translating a and b back into the corresponding individual derivatives, we get exactly the formula we expect, given by the Chain Rule.

Enough bibble-babble! Let's *use* the Chain Rule!

Examples. Compute the derivative of each of the following functions. (Note that, especially at first, it certainly doesn't hurt to *explicitly* break each composition down into its individual pieces, compute $g'(x)$ and $f'(u)$ separately, and *then* put it back together!)

1. $F(x) = \sin(x^2)$

2. $G(x) = \sin^2(x)$

3. $H(t) = \sqrt{e^t}$

4. $J(s) = e^{\sqrt{s^2+1}}$ (Note that you've got a composition of *three* functions here! Use the Chain Rule twice.)

There are certain compositions of functions that come up often enough that it's worthwhile getting used to the form of their derivatives. For instance, consider a *generalized power function*, of the form $(f(x))^n$, for some function f and some real number n. Go ahead and use the Chain Rule to compute the derivative of this sort of function:

$$\frac{d}{dx}\Big((f(x))^n\Big) =$$

Example. Find the equation of the tangent line to the graph of $F(x) = \cos^5(x) = (\cos(x))^5$ at the point $x = \frac{\pi}{4}$.

Another common sort of composition is a *generalized exponential function*, of the form $e^{f(x)}$ for some function f. You should be able to find a simple formula for the derivative $\frac{d}{dx}(e^{f(x)})$ in the space below:

Example. The population of a colony of bacteria is given by the function $g(t) = 10e^{t^3-t}$ for values of t in $[0, 3]$, where t is measured in days. Find the values at which the population obtains either a maximum or a minimum. (Recall that the derivative should equal ____ at these values of t.)

Homework from Section 3.7 (pp. 177-180): numbers 13, 14, 18, 20, 24, 27, 30, 37, 41, 45, 49, 57, 62, 68, 69, 91, and 93. This homework is due on *Friday, October 16th.*