Section 3.3: Products and quotients

At this point we know how to differentiate power functions, exponential functions, and sums or constant multiples of functions whose derivatives we already know. (For example, we can handle all polynomials at this point, but not a whole lot else.)

Our next rule extends the family of functions we can differentiate easily:

______________ **Rule.** Suppose f and g are differentiable at c. Then so is the product $f \cdot g$, and

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = \underline{\hspace{6cm}} \quad .$$

Written in "prime" notation, and surpressing the variable x, we have the following formula:

$$(fg)' = \underline{\hspace{3cm}} \quad .$$

Note. Since addition is a " ______________ " operation, it does matter in which order we put the terms in the righthand side of these equations.

Examples. Find the derivatives of the following functions.

1. $f(x) = (x^2 + 6)(x^3 - 4x + 1)$

2. $H(t) = t^3 e^t$

Example. Find the value of x at which $f(x) = \frac{e^x}{x}$ has a horizontal tangent line. (*Hint:* $f(x) = x^{-1}e^x$...)

Your textbook gives a proof of the Product Rule, on pages 144-145. I think it's more enlightening to think of the rule in terms of areas. If we think of $f(x)$ and $g(x)$ as the width and height, respectively, of a rectangle, we get the following picture:

Then we can "adjust" the rectangle by adding in extra area when we change x by a *little bit*, Δx. The *new* width and height are given, roughly, by $f(x) + \Delta f$ and ________________ . You should modify your picture above to demonstrate this!

Now the change in the *area* corresponds to the change, Δfg, in the product $f(x)g(x)$...that is,

$$\Delta fg \approx \text{ old area } - \text{ new area.}$$

Filling in all of the areas of the little rectangles, we get the following formula:

$$\Delta fg \approx$$

To find $\frac{d}{dx}(fg)$, we now just divide by Δx and take the limit as $\Delta x \to 0$. Finish it off!:

So much for products...what about *quotients*?

______________ **Rule.** Suppose that f and g are differentiable at c and $g(c) \neq 0$. Then $\frac{f}{g}$ is differentiable at c, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \underline{\hspace{4cm}} .$$

Examples. Find the derivatives of the following functions.

1. $g(x) = \frac{x^4}{2x^2+x}$

2. $f(x) = \frac{e^x+5x^2}{2x^3+\sqrt{x}}$

3. $K(x) = \frac{e^x}{x}$ (Note that we already found this derivative in an example above by turning it into a product!)

Example. Find the equation of the tangent line to the graph of $y = \frac{1}{x^2+1}$ at the point $x = 1$.

(More) homework from Section 3.3 (pp. 148-150): numbers 1-30 (even numbers only), 43, 44, 49, and 51. This homework is due on *Friday, October 9th.*