MATH 191, Sections 1 and 3
Calculus I
Fall 2009

Section 3.2, part 1: Derivatives as functions, and some simple derivatives

In the previous section we computed the derivatives of various functions at various points, obtaining in each case a *number* representing the instantaneous rate of change of the function at the given value of the independent variable, which is also the slope of the ______________ line through the graph of the function at the given value of the independent variable.

As our work showed us, computing derivatives can be a cumbersome activity, and it would be great to have a time-saving way of computing many derivatives at once.

This can be done if we think of the derivative of $f(x)$ as a *function* of the variable x; rather than computing $f'(a)$ for a single value of a, we'll compute $f'(x)$ for the *variable* x, using the following formula:

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \underline{\hspace{4cm}} \,.$$

If this limit exists for all x on an interval (a, b), we say that f is ____________________ on that interval.

A note on notation. The notation $\frac{df}{dx}$ is Leibniz's notation for the derivative of f with respect to x, and is meant to remind us of the notation $\frac{\Delta f}{\Delta x}$. Notice that, in fact,

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$

In this way, Leibniz's notation is convenient, clear, and very intuitive. (It's *vastly* superior to Newton's notation, and is one of the reasons that Leibniz's work has gained so much respect throughout the years.)

Examples. Compute $\frac{df}{dx}$ in each case given below.

1. $f(x) = x^2$

2. $f(x) = x^3$

What formula do you suppose might hold for $f'(x)$ if $f(x) = x^n$?

Verification of this formula when n is a natural number. Assume n is a natural number, and let $f(x) = x^n$. Here's a fact that will come in handy:

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}),$$

as can be checked by multiplying the righthand side out.

What's this tell us about the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$? Work it out in the space below!:

Note. The formula is valid for *all* real numbers n and for all x in the domain of $f(x) = x^n$, but we're only able to prove it easily if n is a natural number.

Examples. Compute each of the following derivatives:

1. $\frac{d}{dx}x^5 =$
2. $\frac{d}{dr}r^{100} =$
3. $\frac{d}{dy}y^{-4} =$
4. $\frac{d}{dx}x^{\pi} =$

Here are a couple of other nice properties of the derivative that we can prove in general:

Linearity formulas. Let f and g be differentiable and let c be any constant. $f+g$ and cf are both differentiable functions, and

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \quad \text{and} \quad \frac{d}{dx}(cf) = c\frac{df}{dx}.$$

What do these rules look like in "prime" notation?

Proof of the constant multiple rule. (Your textbook proves the rule for sums.) Let f be differentiable and let c be a constant. Start with the limit definition of the derivative $(cf)'$ and go from there!:

See how easy? We can now compute the derivatives of many functions (like polynomials) very easily:

Examples. Compute the derivative of each of the following functions.

1. $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$

2. $f(x) = 3x - 2\sqrt{x}$

3. $g(x) = \frac{4}{x^2} - x^7 + 7$

4. $h(t) = \frac{1}{3}t^2 + \frac{1}{2}t^2 + t$

Example. Find the points on the graph of $f(x) = x - 3x^2$ where the tangent to the graph is horizontal. (*Hint:* what must be true of $f'(a)$ at such points $x = a$?)

Homework from Section 3.2 (pp. 139-142): numbers 6, 7, 14, 20, 23-34, 39, 53, and 57. This homework is due on *Friday, October 2nd.*