Section 3.11: Related rates of change

In applications, one frequently deals with problems in which more than one quantity is changing with respect to change in yet another. Even if some of the rates of change are unknown, the relationships between the rates can often be exploited in order to determine the unknown values. (The Chain Rule plays a crucial rôle here.) The best way to learn about the method of "related rates" is to do a bunch of examples!

Example. Suppose a sphere's volume is increasing at a rate of 100 cm^3 per second. How fast is the circumference of the sphere changing when the diameter of the sphere has diameter $D = 50$ cm?

Solving this problem requires first understanding what is being asked (that is, *reading* the problem), and then *drawing* a picture or using a visual aid (*introducing notation* where necessary) in order to help us *express* the relevant rates of change in in terms of derivatives. From here, it's a matter of *writing an equation* relating the quantities involved, *differentiating*, and then finally *plugging in* the information given. (The italicized words here summarize the steps your text recommends in solving these problems.

I hope we've all *read* the problem now. In the space below, *draw a diagram* illustrating what's going on:

Now for some notation. Naturally, let's pick V to represent the ____________ of the sphere at a given time, and D to represent its ____________ . We may as well also let C denote the sphere's ________________________ . With these symbols, we can *express* the quantities of interest, and their rates of change, below:

Remember these geometric formulas? You should keep basic formulas like these in mind!

Now we come to the next step in our procedure: which *equation* do you want to *differentiate*? It ought to be one that involves the variable whose rate of change you know, and the variable whose rate of change you seek. Do the work below, remembering that the Chain Rule's going to pop up:

Now that all differentiation is done, we can *substitute* in our values for $\frac{dV}{dt}$ and D in order to solve for the desired value, $\frac{dC}{dt}$:

Voilà! You've done your first related rates problem!

To recap, here's an outline of the procedure we followed above:

1. *Read* the problem carefully, to understand what's being asked and *draw* a diagram or picture or use a visual aid to help
2. *introduce* the relevant quantities, including notation, as well as *expressions* giving the rates in terms of derivatives. Then
3. *write* equations relating the quantities involved. Use the
4. *Chain Rule* to get an equation relating all of the rates, and finally
5. *substitute* in the values you're given. This step must come last: if it's done first, you'll be left dealing with a *particular* instance of the problem which must be solved with *general* computations.

Here are some more

Examples. Follow the procedure as outlined above to solve the following problems.

1. Suppose a conical tank (with its wide base at the top) is 2 meters across at the top and 5 meters high. Suppose further that we are pumping water into the tank at a rate of 2 m^3 per minute. At what rate is the water level rising when the level is 3 meters from the bottom? (Note that you have to use similar triangles find a formula relating the volume V of the water in the tank, and the height h of the water at a particular time. Then using the Chain Rule gives the relationship between the rates of change.)

2. Now suppose that a 3-meter ladder is propped against a wall in such a fashion that the top of the ladder slides downward at a rate of 1 m/s. How fast is the *bottom* of the ladder sliding *away* from the wall when the bottom is 2 meters from the wall?

3. Suppose that a boat is being dragged to the pier by means of a 100 meter cable strung over a pulley which rests 5 meters about the bow of the boat. If the cable is being pulled in at a rate of 1 meter per second, at what rate is the angle between the cable and the horizontal (measured at the bow of the boat) increasing when 87 meters of the cable have been reeled in?

4. Suppose our friend, standing atop a building 10 meters in front of us, lowers a piano to the ground from the building's roof. If the piano is lowered at 1 meter per second, find the rate at which the *angle* between the piano's location and our eyelevel changes when the piano is 10 meters above eyelevel.

   Here's some room for a diagram of what's going on:

And here's some space to solve the problem:

Once you get a bit more adept at solving these problems, you can omit, or combine, the steps above that you don't find incredibly helpful.

Homework from Section 3.11 (pp. 204-207): 1, 5, 7, 12, 15, 26, 28, and 36. This homework is due on *Friday, October 23rd*.