Section 2.6: Key trig limits

Far harder to establish than any of the limits we've considered so far are certain important limits of trigonometric functions. There are two limits we'll focus on today, both of which will prove incredibly important in the work we'll do this semester.

Two Important Limits.

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1 \quad \text{and} \quad \lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}=0.$$

To verify these limits, we'll need the following intuitively obvious fact:

The Squeeze Theorem. Suppose that on some open interval containing the point $x = c$ (but not necessarily *at* $x = c$), we know that

$$\ell(x) \leq f(x) \leq u(x).$$

Suppose we also know that $\lim_{x\to c}\ell(x) = \lim_{x\to c}u(x) = L$. Then $\lim_{x\to c} f(x) =$ ______ as well.

Roughly, f is *squeezed* in between ℓ and u, and since they go to the same value as x approaches c, f is forced to go to that value as well.

Here's some room for a picture of the Squeeze Theorem:

This seemingly simple fact enables us to "tame" pretty wild limits, as in this

Example. Find the limit $\lim_{x\to 0} x\sin(\frac{1}{x})$ using the Squeeze Theorem and the right choices of ℓ and u. (*Note:* we'd be in big trouble if we didn't have the extra x multiplied by $\sin(1/x)$!)

The key to our verification of the important trig limits given above is the following fact, which, when coupled with the Squeeze Theorem, will tell us all we need to know:

Theorem. For all θ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ (except $\theta = 0$), $\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1$.

To prove this we'll need the following figures, each of which denotes a certain area related to the unit circle:

1. What's the area of the first triangle, using the formula $A = \frac{1}{2}bh$ for the area of a triangle?

2. What's the area of the sector shown in the second figure?

3. What's the area of the second triangle, in the third figure?

4. Compare the areas in the figures above...what inequalities arise?

5. Simplify the first inequality as much as you can. What do you learn?

6. Simplify the second equality as much as you can. What do you learn?

We've now shown that $\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1$ on the interval $(-\pi/2, \pi/2)$. Now use the Squeeze Theorem with $\ell(\theta) = \cos(\theta)$ and $u(\theta) = 1$, as $\theta \to 0$. What happens?

Et Voilà! We've proven the first of the trig limits we set out to prove.

The second limit follows easily from the first: let's multiply the top and bottom of $\frac{1-\cos(\theta)}{\theta}$ by the "conjugate" $1+\cos(\theta)$, and simplify as much as we can. The first limit will help at a certain point!

These limits will help us immeasurably in our analysis of trig functions later on. They also give us a means of computing related limits through a "change of variables":

Example. Compute the limit $\lim_{x\to 0} \frac{\sin(4x)}{7x}$. (*Hint:* let $u = 4x$ and change the limit into one involving u instead of x...)

Homework from Section 2.6 (pp. 102-104): numbers 4, 9, 13, 14, 22, 27, and 39. This homework is due on *Friday, September 18th.*