Section 2.4: Continuity

We've already noted that a number of functions have limits that can be evaluated just by *plugging in* values. Roughly speaking, this has happens whenever the function f has no hole, jump, or asymptote at the point we're interested in. We can make this precise with the following

Definition. We say that the function f is ________________ at the point $x = c$ if $\lim_{x\to c} f(x) =$ ____ . If the limit does not exist, or if it is not equal to ____ then we say that function has a ________________ at $x = c$.

Notice that these are precisely the functions for which we can compute limits just by plugging in. Three things have to be true in order for the function f to be ________________ at $x = c$:

1. $\lim_{x\to c} f(x)$ __________ ,
2. $f(c)$ itself __________ , and
3. these two values are __________ .

To understand just what can prevent continuity, let's examine some

Examples. For each function f below, determine the values $x = c$ at which the function is *not* continuous, and indicate for each point *why* there is a discontinuity. (It might not hurt to draw a graph for each function!)

1. $f(x) = \frac{(x-1)(x+2)}{x-1}$

2. $f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 1. \end{cases}$

3. $f(x) = \frac{1}{x^2}$

4. $f(x) = \sin(\frac{1}{x})$

The above discontinuities are of the following types, respectively, at $x = c$:

1. A ____________ *discontinuity* occurs when the function has a "hole" in its graph; these discontinuities can be "removed" by redefining the value of the function at the xs where there are "holes."
2. A ______ *discontinuity* occurs when the function " _______ " from one value to another: note that we cannot "remove" such discontinuities. (However, we can define ____________ *continuity* to handle these cases.)
3. An ____________ *discontinuity* occurs when the function has a vertical asymptote at $x = c$.
4. An ____________ *discontinuity* occurs when the function oscillates periodically infinitely often as we approach $x = c$.

On the other hand, many familiar functions are continuous just about anywhere we would like them to be. Below you should the functions you're familiar with that are continuous *everywhere* on their domains:

If a function is continuous at every point in its domain, it's simply called *continuous*.

Once we know that two functions f and g are continuous, we can build new continuous functions by combining them:

Combining continuous functions. Suppose f and g are both continuous at $x = c$. Then

1. $f(x) \pm g(x)$ is continuous at $x = c$,
2. $kf(x)$ is continuous at $x = c$, for any constant k,
3. $f(x) \cdot g(x)$ is continuous at $x = c$, and
4. $\frac{f(x)}{g(x)}$ is continuous at c provided $g(c) \neq 0$.

Furthermore, if g is continuous at $x = c$ and f is continuous at $g(c)$, then the composition $(f \circ g)(0)$ is continuous at $x = c$ as well, and if f is continuous and one-to-one on its domain, then the inverse function f^{-1} is continuous on *its* domain.

Example. Use the rules above to show *carefully* that $f(x) = \frac{3^x}{1+4^x}$ is a continuous function.

More examples. Decide where each of the following functions are continuous.

1. $g(x) = \sin(x) + x$
2. $h(t) = \frac{\sin(t)}{t^2-4}$
3. $f(x) = \ln(x^2 + 1)$

Homework from Section 2.4 (pp. 90-93): numbers 3, 4, 11, 14, 16, 19, 23, 28, 33, 36, 53, 55, 61, 66, 71, and 78. This homework is due on *Friday, September 18th.*