Section 2.2: Limits, the core concept of calculus

Let's start things off with an

Example. Let $f(x) = \frac{1}{x}$. We are going to compute instantaneous rate of change of f at the value $x = 2$ by first finding the slope of the secant line through the points $(2, f(2))$ and $(2 + h, f(2 + h))$ for a very, very small h and then taking the ________ as h tends to 0.

We begin by setting up a formula for the slope of the secant line and simplifying as much as possible:

Note that, as you've done above, if $h \neq 0$, we can cancel it from the numerator and the denominator of the fraction giving us the slope.

Now comes the $64,000 question: what if we let h becomes very, very close to 0, without actually getting there? What number does your formula for the slope approach?

It's crucial to recall that we can't simply *plug in* $h = 0$, since if $h = 0$ we would not have been able to cancel h in the previous step.

What we've done above is taken the ________ of the function $\frac{1}{4+2h}$ as h goes to 0. ________ are useful any time we need to "sneak up on" a number without actually getting there. Here's a formal

Definition. Suppose that the function f is defined near (but not necessarily at) the value $x = c$. Then we say that the ________ *of* $f(x)$ *as* x *approaches* c is L (written $\lim_{x \to c} f(x) = L$) if we can make $|f(x) - L|$ as small as we'd like to by choosing x close (*but not equal to*) c.

Above, for example, $\lim_{h \to 0} \frac{-1}{4+2h} =$ ______ .

Often we can guess limits by examining a graph, plugging in values, or just straightforward algebra, as in the following

Example. Explain why $\lim_{x\to 2} 3x + 1 = 7$. (*Hint:* We must show that $|3x + 1 - 7|$ can be made as small as we'd like to make it, for x close to 2...start with this absolute value and see if you can make it look like something useful...)

Other times it helps us to consider a graph, as in the following

Example. Guess $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$ by using a graph. Here's some space to sketch what appears to be going on near $\theta = 0$:

Note that even though $f(0)$ is undefined, a limit clearly exists! This is not the case in the following example, showing that limits need not always be defined:

Example. What appears to be going on with $\lim_{x\to 0} \sin(\frac{1}{x})$?

This "infinite oscillation" isn't the only reason a limit can fail to exist. It could be that as we near $x = c$, the function behaves differently on either side of that key value:

Example. Consider the piecewise function $g(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$ What's $\lim_{x\to 0} g(x)$, do you think? (Draw the function's graph in the space below, if that helps you.)

We'd *really* like to make the limits above work "halfway," if possible. Therefore, let's define ____________ *limits* and ____________ *limits* in the obvious fashion:

Definitions. We say that the limit $f(x)$ *as* x *approaches* c *from the* ________ is L (and write ____________) if we can make $|f(x) - L|$ as small as we'd like to by choosing x close to (but not equal to c) and ______ . An entirely analogous definition holds for the limit of $f(x)$ as x approaches c from the ________ .

Collectively these are called ____________ *limits.*

Examples. Get together with the folks in your group to evaluate the following limits in one way or another (plug in values, use graphs, argue algebraically, whatever works!). If a limit does not exist in any fashion, indicate that this is so.

1. $\lim_{t\to -1} t^2 + 1$

2. $\lim_{x\to 2} \frac{(x-2)(x+2)}{x-2}$ (Notice that you can't simply cancel out the $x - 2$ terms, since the resulting function would then be different at the value $x = 2$.)

3. $\lim_{x\to 0} x^{400} + 3x^{50} + 1$

4. $\lim_{\theta\to \pi/4} \tan(\theta)$

5. Let $f(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ x + 1 & \text{if } x > 0. \end{cases}$ Find both $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$. Does $\lim_{x\to 0} f(x)$ exist?

6. $\lim_{x \to 0} \frac{1}{x^2}$

In this last example, even though the limit doesn't technically exist, we'd really like to say something about its behavior. The following definition should not be too shocking:

Definition. We say that $f(x)$ has an ______________ *limit* at $x = c$ if we can make $f(x)$ arbitrarily large (positive or negative) by choosing x close to, but not equal to c. We write $\lim_{x \to c} f(x) =$ _____ or $\lim_{x \to c} f(x) =$ _____ as appropriate. Similar definitions give us ______________ one-sided limits.

Examples. With this definition, what can you say about $\lim_{x \to 0} \frac{1}{x^2}$, above?

What's happening with the function $g(x) = \frac{1}{x}$ at $x = 0$? Go ahead and draw it below, if you'd like to:

Homework from Section 2.2 (pp. 76-79): numbers 3, 7, 8, 11, 19, 22, 27, 37, 38, 45, and 47. This homework is due on *Friday, September 11th.*