MATH 191, SECTIONS 1 AND 3
CALCULUS I
FALL 2009

Section 1.6: Precalculus preliminaries, V (exponential functions and logarithms)

Recall that an __________________ *function* is any function of the form $f(x) = b^x$, for a constant $b > 0$. The following rules help immensely in dealing with exponential functions:

Rules for Exponents. Let $b > 0$. The following are all true, for all real numbers x and y, and all natural numbers n:

1. $b^0 = 1$
2. $b^x \cdot b^y = b^{x+y}$
3. $\frac{b^x}{b^y} = b^{x-y}$
4. $b^{-x} = \frac{1}{b^x}$
5. $(b^x)^y = b^{xy}$
6. $b^{1/n} = \sqrt[n]{b}$

Examples. Use these rules to simplify all of the following expressions as much as possible.

$2^7 \cdot 2^{-3} =$ _________

$16^{-1/2} =$ _________

$256^{3/8} =$ _________

$\frac{(3^2 2^{-1})^5}{4^{-3/2}\sqrt[3]{27}} =$ _________

Recall that there are three "basic" graphs of exponential functions $f(x) = b^x$, depending on whether $0 < b < 1$, $b = 1$, or $b > 1$:

The number $e \approx 2.718281828...$ is a very special number; it is defined to be the quantity such that the graph of $f(x) = e^x$ has slope equal to the value of the function at every point.

As useful as exponential functions are, it's no surprise that their inverses, __________________ *functions*, are just as useful.

Definition. If $b > 0$ (and $b \neq 1$), the __________________ of x *with base* b, denoted $\log_b(x)$, is defined by

$$b^{\log_b(x)} = x \quad \text{and} \quad \log_b(b^x) = x.$$

That is, $\log_b(x)$ is the number to which we must raise b to obtain x, and the functions b^x and $\log_b(x)$ are inverses to one another.

From the rules for exponents come

Rules for Logarithms. Let $b > 0$. Then the following are all true, for any x, y, and n:

1. $\log_b(1) = 0$ and $\log_b(b) = 1$
2. $\log_b(xy) = \log_b(x) + \log_b(y)$
3. $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
4. $\log_b(x^n) = n \log_b(x)$

Note that the ______________ *logarithm* of x, denoted $\ln(x)$, is the logarithm with base $b = e$. (Thus $\ln(x) = \log_e(x)$.)

Examples. Simplify the following expressions as much as you can:

$\ln(e^7 e^2) =$ __________

$2^{\log_2(6)} =$ __________

$4^{\log_2(6)} =$ __________

$\log_6(9) + \log_6(4) =$ __________

More examples! Use the rules for exponents to solve for the unknown variable.

1. $2^{3x+1} = 2^5$

2. $(b^2)^{x+1} = b^{-6}$

Finally, let's see how exponential and logarithmic functions might help us to *model* a given problem from outside of pure mathematics.

Application. We are given a sample of the element bismuth to test, and we're asked to identify the isotope by computing the *radioactive half-life* of the sample. (Recall that this is the time it takes half of a given sample to decay.) All we're given is the mass, $m(t)$, of the bismuth in the sample at two points in time: $m(0) = 100$ g and $m(60) = 12.37$ g, where time is measured in minutes.

Step 1. Knowing that exponential functions describe radioactive decay, let's model the mass as an exponential function: $m(t) = m_0 e^{kt}$ for some unknown constants m_0 and k.

Use the first measurement, $m(0) = 100$ g, to find the value of m_0.

Step 2. Now use the second measurement, $m(60) = 12.37$ g, to find a value for k.

In summary, our formula for $m(t)$ is now $m(t) =$ ____________________ .

Step 3. To find the half-life of the sample, we must determine at what time half of the original sample remains. That is, we solve the equation $m(t) = \frac{100}{2} = 50$. Do this in the space below.

Step 4. Now let's look on-line to see which isotope of bismuth has a half-life closest to the one we've computed. Which isotope is it?

Homework from Section 1.6 (pp. 52-53): numbers 1, 3, 5, 8, 12, 13, 16, 17, 20, 23, 28, 35, and 36. This homework is due on *Friday, September 4th.*