Section 1.5: Precalculus preliminaries, IV (inverse functions)

Generally we think of a function $y = f(x)$ as rule for deciding what output, y, comes when a certain input, x, is plugged into f. However, certain functions $f(x)$ are such that we may be able to "follow the function backwards," obtaining from a given value y the value of x that one must plug *into* f to get y as *output*.

Examples. If we know $y = f(x) = x^3$, what value of x must be plugged in to obtain $y = 8$ as output? $y = 27$? How about y in general?

What if $y = 3x + 1$: what value of x must be plugged in to obtain the value a given value $y = 10$ as output? What about $y = -2$? What about y in general?

What most likely did to solve the problems above was find the *inverse* of the given function f:

Definition. If f is a function with domain D and range R, then f^{-1} (if it exists) is the function with domain R and range D such that

$$f^{-1}(f(x)) = x \text{ for all } x \text{ in } D, \text{ and } f(f^{-1}(y)) = y \text{ for all } y \text{ in } R.$$

All this is saying is that f and f^{-1} "undo" each other: doing one and then the other (in either order) gets us back where we started.

Example. Find the inverse of the function $y = \sqrt[3]{x} - 2$.

As in the previous example, we can always fine the inverse (if it exists) by

1. Writing $y = f(x)$,
2. solving this equation for x in terms of y, and
3. switching the roles of x and y in the resulting equation.

Example. Find $g^{-1}(x)$ if $g(x) = \frac{1}{7x-3}$. (What's the domain of g?)

Many functions, even simple, common ones, can *not* be inverted.

Example. Why can we not invert the function $y = x^2$? (*Hint:* for a given y we need to find a *single* x which when plugged into f gives y...)

Definition. A function f is called ____________________ if any time $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. That is, different inputs give different outputs.

It's easy to see that if f is ____________________ then $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Examples.

1. Is the function $f(x) = x^2$ one-to-one? Why or why not?

2. Is the function $g(x) = 2x + 4$ one-to-one? Can you prove your claim true?

3. Is the general linear function $h(x) = mx + b$ always one-to-one? Under what conditions is it?

4. Is the function $f(x) = x^2$ one-to-one if we restrict its domain to $[0, \infty)$? If so, what's the inverse function, $f^{-1}(x)$?

Testing whether a function is one-to-one graphically is made easy by the ________________ **Line Test**: If every ________________ line intersects the graph of the function $y = f(x)$ at most once, then f is one-to-one.

Here's some space for you to draw a graph illustrating this proposition; notice that if the Horizontal Line Test fails, it's because the same y value comes as output for two *different* x-values, which is precisely what it means for the function to fail to be one-to-one!

We should mention before finishing that just as the function $y = x^2$ can be made one-to-one by restricting its domain, so can the functions $y = \cos(x)$, $y = \sin(x)$, and $y = \tan(x)$. The following definitions of the *inverse trig functions* come in handy:

Definitions.

1. If x is in $[-1, 1]$, the function $\theta = \sin^{-1}(x)$ is the unique angle in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(\theta) = x$.
2. If x is in $[-1, 1]$, the function $\theta = \cos^{-1}(x)$ is the unique angle in the interval $[0, \pi]$ such that $\cos(\theta) = x$.
3. If x is any real number, the function $\theta = \tan^{-1}(x)$ is the unique angle in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan(\theta) = x$.

Examples. Compute each of the following values:

$$\cos^{-1}(1) = \underline{\hspace{2cm}} \qquad \tan^{-1}(1) = \underline{\hspace{2cm}}$$

$$\sin^{-1}(\frac{\sqrt{2}}{2}) = \underline{\hspace{2cm}} \qquad \cos^{-1}(-\frac{1}{2}) = \underline{\hspace{2cm}}$$

Homework from Section 1.5 (pp. 42-44): numbers 2, 13, 14, 16, 24, and 27. This homework is due on *Friday, September 4th.*