MATH 191, SECTIONS 1 AND 3
CALCULUS I
FALL 2009

Practice Exam 2: Solutions

1. (35 points total; 7 points each) Find the derivative of each of the following functions. You may use any shortcut formula you would like.

   (a) $f(x) = x\cos(x)$
   Here we need the Product Rule:
   $$f'(x) = 1 \cdot \cos(x) + (-\sin(x)) \cdot x = \cos(x) - x\sin(x).$$

   (b) $g(t) = \frac{\tan(t)}{e^{t^2}}$
   Now it's all about the Quotient Rule, using the Chain Rule to differentiate the bottom as needed:
   $$\frac{\sec^2(x) \cdot e^{t^2} - e^{t^2} 2t \cdot \tan(t)}{(e^{t^2})^2}.$$

   (c) $H(x) = \sin(\cos(\tan(x)))$
   Now we need the Chain Rule twice, since we have a function of a function of a function:
   $$H'(x) = \cos(\cos(\tan(x))) \cdot -\sin(\tan(x)) \cdot \sec^2(x).$$

   (d) $F(z) = \ln(z^2) + ze^z$
   We need the Chain Rule for the first term, and the Product Rule for the second:
   $$\frac{1}{z^2} \cdot 2z + (1 \cdot e^z + e^z \cdot z) = \frac{2}{z} + e^z(1+z).$$

   (e) $f(x) = \sec(5^x)$ (You can use the formula $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$.)
   Here we need the Chain Rule, recalling the derivative of a general exponential for the derivative of the inside:
   $$\sec(5^x)\tan(5^x) \cdot \ln(5)5^x.$$

2. (10 points) Find the equation of the tangent line to the graph of the expression $y^3 = (x+y)^2$ at the point $(0,1)$.

   We're given a point on the line we seek, namely $(0,1)$ itself. Thus all we need to find the tangent line is the slope, given by the derivative $y' = \frac{dy}{dx}$. To find this, we differentiate implicitly:
   $$\begin{aligned} 3y^2y' = 2(x+y)(1+y') &\Rightarrow 3y^2y' = 2x + 2y + 2xy' + 2yy' \\ &\Rightarrow (3y^2 - 2x - 2y)y' = 2x + 2y \\ &\Rightarrow y' = \tfrac{2x+2y}{3y^2-2x-2y}. \end{aligned}$$

To find y' at $(0,1)$, we just plug in:

$$y'(0,1) = \frac{0+2}{3-0-2} = \frac{2}{1} = 1.$$

At last, we use this value to find the formula we seek:

$$y - y_1 = m(x - x_1) \Rightarrow y - 1 = 1(x-0) \Rightarrow y = x+1.$$

3. (10 points) Compute $\frac{d}{dx}(\frac{1}{x^2})$ using the *definition* of the derivative (and *not* shortcuts!)

   The general formula, of course, is $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. In this case, we get

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}}{h}.$$

   Combining denominators and canceling, we get

$$\lim_{h\to 0} \frac{-2xh-h^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{h(-2x-h)}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} = \frac{-2x}{x^2\cdot x^2} = \frac{-2}{x^3},$$

   where in the second-to-last step we've plugged in $h=0$ safely. Notice that this is the formula we expected from the Power Rule: $f'(x) = -2x^{-3}$!

4. (10 points) Find the all values of x at which the tangent line to the graph of the function $g(x) = x^2e^x$ is horizontal.

   It's just a matter of computing $g'(x)$ (using the Product Rule) and setting it equal to 0 to see which xs result:

$$g'(x) = 2xe^x + x^2e^2 = e^x x(2+x),$$

   so $g'(x) = 0$ implies that either $x = 0$ or $2 + x = 0 \Rightarrow x = -2$, since $e^x \neq 0$ for all x.

5. (10 points) Use logarithmic differentiation to find the derivative of $f(x) = x^{\sin(x)}$ at $x = \pi$.

   We start by writing $y = x^{\sin(x)}$, and then take the natural log of both sides:

$$\ln(y) = \ln(x^{\sin(x)}) = \sin(x)\ln(x).$$

   Now it's time to differentiate, implicitly, using the Chain Rule on the left and the Product Rule on the right:

$$\frac{1}{y}y' = \cos(x)\ln(x) + \frac{\sin(x)}{x}.$$

   To isolate the desired derivative, we merely have to multiply by y:

$$\frac{dy}{dx} = y' = \Big(\cos(x)\ln(x) + \frac{\sin(x)}{x}\Big)x^{\sin(x)}.$$

   Now we can plug in $x = \pi$:

$$\Big(\cos(\pi)\ln(\pi) + \frac{\sin(\pi)}{\pi}\Big)\pi^{\sin(\pi)} = \Big(-1\cdot\ln(\pi) + \frac{0}{\pi}\Big)\pi^0 = -\ln(\pi).$$

6. (15 points) Suppose that water is being drained from a conical tank with its vertex pointing downward. The tank has a base radius of 2 meters and a height of 5 meters. If the water is drained at a rate of 1 m^3 per minute, how fast is the water level falling when the water is 3 meters deep in the tank? (*Hint:* use similar triangles to determine a relationship between radii and heights...)

   First, let's identify the quantities we're interested in, and give them names. We're going to be working with the *volume*, V, of the water, the *height*, h, of the water level, and time, t. What information are we given, in terms of this notation? We're told that $\frac{dV}{dt} = -1$, where we use a negative value to indicate that the volume of the water is *decreasing* at the given rate. What is it that we're supposed to determine? We'd like to know $\frac{dh}{dt}$, when $h = 3$.

   Since we'd like to find a relationship between $\frac{dV}{dt}$ and $\frac{dh}{dt}$, we first seek a relationship between $V = h$. The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. The problem? We don't yet know anything about r.

   That's okay! Since the water and the tank itself share an angle (at the tank's bottom), we know their vertical cross-sections are similar triangles, so the corresponding sides are proportional in length. Comparing the radius of the tank with its height, and the corresponding measures of the water, we see

$$\frac{1}{5} = \frac{r}{h} \Rightarrow r = \frac{1}{5}h = \frac{h}{5}.$$

   Substituting this in for r in the formula for V, we see

$$V = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h = \frac{\pi h^3}{75}.$$

   At last, we're ready to differentiate! Taking the t-derivatives of both sides and using the Chain Rule, we obtain

$$\frac{dV}{dt} = \frac{\pi}{75}3h^2\frac{dh}{dt}.$$

   Now plug in our known quantities, $\frac{dV}{dt} = -1$ and $h = 3$:

$$-1 = \frac{\pi}{75}3(3)^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{25}{9\pi} \text{ m/min}.$$

7. (10 points) True or false: if a function is continuous, then it is differentiable. (Please explain your answer carefully, using examples when appropriate.)

   This is **FALSE** in general: the function $f(x) = |x|$ is continuous at $x = 0$, but it is *not* differentiable there, since at $x = 0$ the derivative $f'(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0 \end{cases}$ has a jump discontinuity. Another example if the function $g(x) = x^{1/3}$, which is continuous at $x = 0$ but not differentiable there, as in this case, the derivative $g'(x) = \frac{1}{3}x^{-2/3}$ has a vertical asymptote at $x = 0$.