MATH 191, Sections 1 and 3
Calculus I
Fall 2009

Practice Exam 1: Solutions

1. (24 points total; 8 points each) Indicate whether each statement is true or false, and give a *brief* (roughly one- or two-sentence) explanation for your answer.

   (a) The average rate of change of the function $f(x)$ between $x = x_0$ and $x = x_1$ is the same as the slope of the tangent line to f's graph through the point $(x_0, f(x_0))$.
   FALSE. The average rate of change is the same as the slope of the *secant* line to the graph between the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

   (b) If $\lim_{x\to c} f(x)$ and $f(c)$ both exist, then the function f is continuous at $x = c$.
   FALSE. It might be the case that the function is continuous, but one more thing *must* be true: the two values must be *equal* to each other.

   (c) The Intermediate Value Theorem can be applied to the function $f(x) = \sin(x)$ on the interval $[-\frac{\pi}{3}, \frac{\pi}{2}]$.
   TRUE. All that the IVT requires is that the function be continuous on the interval given, and indeed this is the case with the function $\sin(x)$.

2. (24 points total; 8 points each) Evaluate each of the following limits. If a limit does not exist (either as a real number or as an infinite quantity), explain briefly why this is the case.

   (a) $\lim_{x\to 0} \frac{\cos(x)}{x+1}$
   This limit is 1:
   $$\lim_{x\to 0} \frac{\cos(x)}{x+1} = \frac{\lim_{x\to 0}\cos(x)}{\lim_{x\to 0} x+1} = \frac{1}{1} = 1.$$

   (b) $\lim_{t\to -2} \frac{t^2+3t+2}{t+2}$
   The numerator factors, and then we cancel (recalling that if $t \neq -2$, $t+2 \neq 0$, and this term can in fact be cancelled):
   $$\lim_{t\to -2} \frac{t^2+3t+1}{t+2} = \lim_{t\to -2} \frac{(t+1)(t+2)}{t+2} = \lim_{t\to -2} t+1 = -1.$$

   (c) $\lim_{\theta\to 0} \frac{\sin(5\theta)}{3\theta}$
   Let's let $u = 5\theta$, so that $3\theta = \frac{3\cdot 5\theta}{5} = \frac{3}{5}u$. Then, noting that u approaches 0 precisely when θ does so,
   $$\lim_{\theta\to 0} \frac{\sin(5\theta)}{3\theta} = \lim_{u\to 0} \frac{\sin(u)}{\frac{3}{5}u} = \lim_{u\to 0} \frac{5}{3}\frac{\sin(u)}{u} = \frac{5}{3},$$
   since $\lim_{u\to 0} \frac{\sin(u)}{u} = 1$.

3. (8 points) Briefly desribe the purpose of both the Vertical Line Test and the Horizontal Line Test, taking care to describe how the two tests differ.

   The *Vertical* Line Test is used in order to decide whether or not a graph represents the graph of a function, whereas the *Horizontal* Line Test is used to determine whether or not a function is one-to-one based upon its graph. Strictly speaking, we can't even use the HLT without first having applied the VLT, since we only talk about *functions* being one-to-one: VLT must be used first to determine whether the graph shows a function in the first place.

4. (8 points) Draw the graph of a *single* function f with *all* of the following properties:

   (a) f has a jump discontinuity at $x = 3$,

   (b) both one-sided limits exist for f at $x = 2$, but the ordinary limit $\lim_{x\to 2} f(x)$ does not exist,

   (c) $\lim_{x\to 0} f(x)$ exists but the function is not continuous there, and

   (d) $\lim_{x\to -2^+} = \infty$.
   There are obviously infinitely many right answers to this question!

5. (12 points total; 6 points each) Suppose that a kumquat is dropped from the top of the Willis (formerly Sears) Tower and that it falls $s(t) = 4.9t^2 - 1.2t$ meters after t seconds have passed.

   (a) Find the average velocity of the kumquat on the time interval $[0, 2]$.
   We simply use the formula for the average rate of change of the function s on the given interval:
   $$\frac{\Delta s}{\Delta t} = \frac{s(2) - s(0)}{2 - 0} = \frac{(19.6 - 2.4) - 0}{2} = \frac{17.2}{2} = 8.6.$$
   Thus our answer is 8.6 m/s.

   (b) Find the average velocity of the kumquat on the time interval $[2, 4]$.
   Now we use the same formula, on a different interval:
   $$\frac{\Delta s}{\Delta t} = \frac{s(4) - s(2)}{4 - 2} = \frac{(78.4 - 4.8) - (19.6 - 2.4)}{2} = \frac{73.6 - 17.2}{2} = \frac{56.4}{2} = 28.2.$$
   Thus our answer is 28.2 m/s.

6. (10 points total; 5 points each) Let $f(x) = \sqrt{x}$ and $g(x) = x + 1$.

   (a) Find a formula for $f \circ g$ and give its domain.
   Just using the definition of composition, we have
   $$(f \circ g)(x) = f(g(x)) = f(x + 1) = \sqrt{x + 1}.$$
   For this to be defined, we need $x + 1 \geq 0$, or $x \geq -1$. Thus the domain of $f \circ g$ is $[-1, \infty)$.

(b) Find a formula for $g \circ f$ and give its domain.

Again, with the definition of composition we have

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 1.$$

Now we require $x \geq 0$, so the domain of $g \circ f$ is $[0, \infty)$.

7. (14 points total; 7 points each) Let $f(x) = (x-2)^2 + 1$.

(a) Sketch a graph of the function f and explain in one sentence how to obtain this graph without using a calculator.

Your graph should be the ordinary parabola $y = x^2$, shifted to the right by 2 units, and up by 1 unit.

(b) Explain in one sentence how you can tell that f is one-to-one on the interval $[2, \infty)$, and find its inverse, $f^{-1}(x)$, on that interval.

You can apply the Horizontal Line Test to the graph you drew for (a), noting that if you eliminate the left half of the parabola (the part *not* lying over the interval $[2, \infty)$) the resulting graph passes the HLT. Thus the function is one-to-one on the restricted domain. (Alternatively, you could show directly that f is one-to-one by noting that if $f(x_1) = f(x_2)$, then

$$(x_1 - 2)^2 + 1 = (x_2 - 2)^2 + 1 \Rightarrow (x_1 - 2)^2 = (x_2 - 2)^2 \Rightarrow x_1 - 2 = x_2 - 2 \Rightarrow x_1 = x_2,$$

where the second-to-last equality holds because we've restricted the function's domain.)

To find the inverse, we let $y = (x-2)^2 + 1$ and solve for x:

$$y = (x-2)^2 + 1 \Rightarrow y - 1 = (x-2)^2 \Rightarrow \sqrt{y-1} = x - 2 \Rightarrow \sqrt{y-1} + 2 = x.$$

Switching the roles of x and y, we obtain $f^{-1}(x) = \sqrt{x-1} + 2$.