Application 2: Population dynamics

Calculus comes in handy when we're asked to analyze a number of problems from the natural and life sciences, including biology. One of the easiest applications in the latter field deals with *population dynamics*, the study of how populations of organisms change and interact over time.

A very simple model for population growth is the exponential one, in which the rate of change of the population is proportional to the population itself. This gives rise to the following simple differential equation:

$$P'(t) = kP(t),$$

where $P(t)$ is the population at time t and r is a constant giving the relative growth rate of the popluation.

1. Show that for any constant C, the function $P(t) = Ce^{rt}$ is a solution to the differential equation above.

2. Suppose we know that the population at time $t = 0$ is some fixed number p_0. Find the solution to the differential equation above which satisfies the initial condition $P(0) = p_0$.

3. There are a number of problems with this very simplistic model. Can you name a couple? (*Hint:* it might be worth looking at a graph of this $P(t)$.)

In 1838 the Belgian scientist Pierre-François Verhulst proposed a slightly more complicated differential equation in order to more accurately describe population growth:

$$P'(t) = rP(t)\Big(1 - \frac{P(t)}{K}\Big),$$

r is a constant indicating the relative rate of growth and K is a constant called the *carrying capacity* of the population. (This is essentially the largest population the environment can sustain.)

4. Speaking entirely *qualitatively*, what happens to the term in parentheses as the population $P(t)$ is very, very, *very* close to K (but still less than K)? What does this say about the rate of change of the population?

5. Consider the highly simplified case in which $K = 1$ and $r = 1$. Write down the equation to which the one above simplifies in this case:

6. Show that the function $P(t) = \frac{e^t}{1+e^t}$ is a solution to the equation you wrote down in (5).

7. Plot the graph of the function $P(t)$ given in (6). Why does this seem like a more realistic model for growth?