Problem Sheet 4

Definitions. Let $G = (V, E, \phi)$ and $G' = (V', E', \phi')$ be (undirected) graphs. A *homomorphism* (sometimes called a *graph homomorphism*) $f : G \to G'$ is an ordered pair (f_1, f_2) such that $f_1 : V \to V'$ and $f_2 : E \to E'$ are functions satisfying

$$\phi(e) = \{u, v\} \Rightarrow \phi'(f_2(e)) = \{f_1(u), f_1(v)\}.$$

If both f_1 and f_2 are bijections, we say that f is a (*graph*) *isomorphism* between G and G', and that G and G' are *isomorphic*, denoted $G \cong G'$. If $f : G \to G$ is an isomorphism, we call f an *automorphism*.

Problem 32. List explicitly (that is, give all values of f_1 and f_2 for) three homomorphisms f from

$$P_3 = (\{u_1, u_2, u_3\}, \{e_1, e_2\}, \{(e_1, \{u_1, u_2\}), (e_2, \{u_2, u_3\})\})$$

to

$$C_4 = (\{v_1, v_2, v_3, v_4\}, \{E_1, E_2, E_3, E_4\}, \{(E_1, \{v_1, v_2\}), (E_2, \{v_2, v_3\}), (E_3, \{v_3, v_4\}), (E_4, \{v_4, v_1\})\}).$$

Problem 33. Determine how many homomorphisms there are from P_3 to K_4, and give one of them explicitly.

Problem 34. Determine how many homomorphisms there are from K_4 to P_3, and give one of them explicitly.

Problem 35. Determine how many automorphisms there are from K_4 to itself, and give one of them explicitly.

Problem 36. Prove carefully that the relation "$\cong$" is an equivalence relation on graphs.

Problem 37. Suppose $G \cong G'$ and both graphs are finite. Prove that if G has n vertices with degree d, then G' has n vertices of degree d. That is, there is a bijection between the respective sets of vertices of degree d. (The finiteness condition is not necessary here, but it makes the proof slightly easier.)

Definition. Let $G = (V, E, \phi)$ and $G' = (V', E', \phi')$ be graphs where $V \cap V' = \emptyset$ and $E \cap E' = \emptyset$. The *disjoint union* of G and G', denoted either $G \cup G'$ or $G \sqcup G'$, is the graph $(V \cup V', E \cup E', \Phi)$, where Φ is defined by $\Phi(e) = \phi(e)$ if $e \in E$ and $\Phi(e) = \phi'(e)$ if $e \in E'$.

Definition. Let $G = (V, E, \phi)$ and $G' = (V', E', \phi')$ be graphs. The *product* of G and G', denoted $G \times G'$, is the graph $(V_\times, E_\times, \phi_\times)$ where $V_\times = V \times V'$ is the Cartesian product of V and V', and $E_\times$ and $\phi_\times$ are defined by including an edge $\{(u, v), (u', v')\}$ whenever either of the following holds:

1. $u = u'$ and $\{v, v'\} \in E'$, or

2. $v = v'$ and $\{u, u'\} \in E$.

Definition. Let $G = (V, E, \phi)$ be a graph. The *line graph* of G, denoted $L(G)$, is the graph (V_L, E_L, ϕ_L), where $V_L = E$ and E_L and ϕ_L are defined by including an edge $\{e_1, e_2\}$ whenever $\phi(e_1) \cap \phi(e_2) \neq \emptyset$ in G (that is, whenever e_1 and e_2 share an endpoint in G).

Problem 38. Let $G = K_4$ and $G' = P_3$. Draw and label $G \times G'$, $L(G)$, and $L(G')$.

Problem 39. Suppose G has order n and size m. What can you say about the order and size of $G \times G$? What about the order and size of $L(G)$?

Problem 40. Find infinitely many graphs, each of which is isomorphic to its own line graph.

Problem 41. Show that the collection of graphs contains a *terminal object*, that is, a graph G_0 such that for *any* graph G, there is a homomorphism $f : G \to G_0$. (*Hint:* find such a graph G_0 and prove that it works.)