Problem Sheet 3

Definitions. A *directed graph* (also called a *digraph*) is an ordered triple $G = (V, E, \eta)$ satisfying

1. $V \neq \emptyset$,
2. $V \cap E = \emptyset$, and
3. $\eta : E \to V \times V$ is a function.

Notice that η (the Greek letter "eta") gives as output an *ordered pair*: it matters which vertex comes first!

If $\eta(e) = (u, v)$, we call u the *initial endpoint* of e, and v the *terminal endpoint* of e. If $\eta(e) = (u, u)$, e is called a *directed loop.* If $\eta(e) = \eta(e')$, we call e and e' *parallel edges.*

Convention. Directed graphs are represented pictorially in much that same way as are ordinary graphs, only we take care to draw each edge as an arrow, pointing *from* its initial vertex, *to* its terminal vertex.

Problem 25. Draw the digraph

$$G = \Big(\{a, b, c\}, \{e, f, g, h\}, \{(e, (a, a)), (f, (a, b)), (g, (b, a)), (h, (a, c))\}\Big).$$

Problem 26. Describe a way of using digraphs to model family trees, describing your model carefully (*i.e.*, what are the vertices? What are the edges? *Etc.*)

Problem 27. Let's agree to say that 1 is "below" 2 because 1 evenly divides 2, 2 is "below" 4 because 2 evenly divides 4, 5 is "below" 15 because 5 evenly divides 15, and so on. In general, m is "below" n if m evenly divides n, for any natural numbers m and n. Describe a way of using a digraph to model the relationship "below" and draw the digraph indicating this relationship for the numbers $\{1, 2, 3, ..., 20\}$. (You do not need to prove anything formally, but your model should be explained clearly.)

Definitions. Given a digraph (V, E, η), the *underlying graph* of G is the ordinary (*undirected*) graph (V, E, ϕ) with identical vertex and edge sets, in which the function ϕ is defined by $\phi(e) = \{u, v\}$ whenever $\eta(e) = (u, v)$. (That is, ϕ simply "forgets" that the order matters in crossing e from u to v.) A digraph is called *simple* if its underlying graph is simple.

For any $n \geq 1$, the *directed cycle* on n vertices is the digraph (V, E, η) with $V = \{v_1, v_2, ..., v_n\}$, $E = \{e_1, e_2, ..., e_n\}$, and $\eta(e_i) = (v_i, v_{i+1})$, where arithmetic is performed modulo n. (That is, $\eta(e_n) = (v_n, v_1)$.) The *undirected cycle* (or simply *cycle*) on n vertices, C_n, is the underlying graph of this digraph.

Problem 28. Draw C_n for $1 \leq n \leq 5$. For which n is C_n simple?

Definitions. Let G be a digraph and let $v \in V(G)$. The *indegree* of v, denoted $d^-(v)$, is defined by $d^-(v) = |\{e \in E \mid \eta(e) = (x, v) \text{ for some } x \in V\}|$. The *outdegree* of v, denoted $d^+(v)$, is defined by $d^+(v) = |\{e \in E \mid \eta(e) = (v, x) \text{ for some } x \in V\}|$. The *inneighborhood* of v, denoted $N^-(v)$, is the set $\{x \in V \mid \eta(e) = (x, v) \text{ for some } e \in E\}$. The *outneighborhood* of v, denoted $N^+(v)$, is the set $\{x \in V \mid \eta(e) = (v, x) \text{ for some } e \in E\}$. The digraph G is called *balanced* if for every $v \in V(G)$, $d^-(v) = d^+(v)$.

Problem 29. Draw a balanced digraph of order 5 whose underlying graph is 2-regular.

Problem 30. Draw a balanced digraph of order 6 whose underlying graph is 4-regular.

Problem 31. Prove that for any digraph G,

$$\sum_{v \in V(G)} d^-(v) = \sum_{v \in V(G)} d^+(v) = |E(G)|.$$