Problem Sheet 2

Definition. Let G be a graph, and let $v \in V(G)$. The *neighborhood* of v, denoted $N(v)$ (or $N_G(v)$ if we need to be clear about which graph we're talking about) is the collection

$$\{u \in V(G) \mid \phi(e) = \{u, v\} \text{ for some } e \in E(G)\}.$$

The *closed neighborhood* of v, denoted by $N[v]$ or $N_G[v]$, is defined to be $N(v) \cup \{v\}$.

Problem 17. Let G be a graph, and let $v \in V(G)$. What relationships hold between $d_G(v)$, $N_G(v)$, and $N_G[v]$?

Definitions. Let $G = (V, E, \phi)$ and $G' = (V', E', \phi')$ be graphs. We say that G' is a *subgraph* of G, denoted $G' \leq G$, if the following conditions are met:

1. $V' \subseteq V$,
2. $E' \subseteq E$, and
3. $\phi'(e') = \phi(e')$ for all $e' \in E'$.

If G is a graph and $W \subseteq V(G)$ is a subset of its vertices, we define the *subgraph induced by* W, $G[W]$, to be the subgraph of G with vertex set W, edge set consisting of those edges *all* of whose endpoints are in W, and endpoint function agreeing with G's endpoint function.

If G is a graph and $F \subseteq E(G)$ is a subset of its edges, we define the *subgraph induced by* F, $G[F]$, to be the subgraph of G with edge set F, vertex set consisting of those vertices that are endpoints of some edge in F, and endpoint function agreeing with G's endpoint function.

If $W = \{w_1, w_2, ..., w_n\}$ is a given set of vertices, we often abuse notation and write $G[W]$ by $G[w_1, w_2, ..., w_n]$ instead of $G[\{w_1, w_2, ..., w_n\}]$, to avoid using too many brackets and parentheses. The same holds for edge-induced subgraphs.

Problem 18. Let $G = (V, E, \phi)$ be given by $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_7\}$, and

$$\phi = \Big\{(e_1, \{v_2, v_2\}), (e_2, \{v_2, v_4\}), (e_3, \{v_1, v_2\}), (e_4, \{v_1, v_3\}), (e_5, \{v_1, v_3\}), (e_6, \{v_3, v_4\}), (e_7, \{v_4, v_5\})\Big\}.$$

Draw this graph, and draw the subgraph induced by the vertex set $W = \{v_1, v_2, v_3\}$.

Problem 19. With G as in Problem 18, draw the subgraph of G induced by the edge set $F = \{e_1, e_2, e_4\}$.

Problem 20. Describe the simple graph G of smallest order having the property that *any* simple graph G' of order n appears as a subgraph of G.

Definition. Let $n \geq 1$. The *path graph* (or simply *path*) on n vertices, denoted P_n, is the graph (V, E, ϕ) with $V = \{v_1, v_2, ..., v_n\}$, $E = \{e_1, e_2, ..., e_{n-1}\}$, and

$$\phi = \Big\{(e_1, \{v_1, v_2\}), (e_2, \{v_2, v_3\}), ..., (e_{n-1}, \{v_{n-1}, v_n\})\Big\}.$$

We say that P_n has *length* $n - 1$.

Problem 21. How many times does the graph P_3 appear as a subgraph of the complete graph K_4?

Problem 22. How many times does the graph P_3 appear as an *induced* subgraph of K_4? (That is, how many times does P_3 appear as a subgraph of the form $G[v_1, v_2, v_3]$?)

Problem 23. Prove that if $G' \leq G$ and G is simple, then so is G'.

Problem 24. Prove that the subgraph relation $\leq$ is a *partial order relation* on the collection of all graphs. That is, show that it is *reflexive*, *antisymmetric*, and *transitive*. (Ask me if you don't recall what these terms mean.)