Problem Sheet 1

Definitions. A *graph* G is an ordered triple (V, E, ϕ), satisfying

1. V is a nonempty set,
2. $V \cap E = \emptyset$ (so V and E are disjoint), and
3. $\phi : E \to \mathcal{P}(V)$ is a function such that $|\phi(e)| \in \{1, 2\}$ for all $e \in E$ (that is, there are either one or two vertices in the image of e).

The elements of the set V are called the *vertices* of G, and the elements of E are called the *edges* of G. For each edge $e \in E$, the vertices in $\phi(e)$ are called the *endpoints* of e. If $|\phi(e)| = 1$, e is called a *loop*. When we need to talk about the set V (respectively, E), we often write it as $V(G)$ (respectively, $E(G)$), to remind ourselves which graph it comes from. The function ϕ can be denoted ϕ_G for a similar purpose.

Conventions. We typically denote graphs pictorially. Given a graph G, we draw a dot for each vertex in $V(G)$, and for each $e \in E(G)$ we draw an arc connecting the dots representing the vertices in $\phi(e)$. (If e is a loop, we may draw an actual loop beginning at $\phi(e)$ and ending back there as well.) Although sometimes it's nice to make all of the arcs representing edges *straight line segments*, this is not necessary, and in some cases can be misleading. Moreover, arcs representing edges may cross one another; at such crossings we take care not to draw a dot, lest we think there's a vertex there when there really isn't!

We will also often abbreviate $\phi(e) = \{u, v\}$ to $e = \{u, v\}$, suppressing the role of ϕ as long as this endpoint function is understood.

Whenever we say "draw the graph G" we take this to mean "draw a pictorial representation of the graph G," as described above.

Problem 1. Draw the graph $G = (V, E, \phi)$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, and $\phi = \Big\{(e_1, \{v_1, v_5\}), (e_2, \{v_1, v_2\}), (e_3, \{v_2\}), (e_4, \{v_3, v_5\}), (e_5, \{v_1, v_3\})\Big\}$. (Note the way in which we've written ϕ as a set of ordered pairs, recalling the definition of a function as a relation!)

Definitions. Given a graph G, the *order* of G is $|V(G)|$, and the *size* of G is $|E(G)|$.

Problem 2. What is the smallest order a graph can have? Draw a graph with such order, and give the triple (V, E, ϕ) corresponding to the graph you draw.

Problem 3. What is the smallest size a graph can have? Draw three graphs with such size, such that each of the graphs is fundamentally different from one another in some obvious fashion (we'll make this "fundamental difference" precise later). By the way, a graph with this smallest possible size is called a *null graph*.

Problem 4. Draw as many fundamentally different graphs as you can, each having order 4 and size 3, also writing each as a triple.

Definitions. Let $G = (V, E, \phi)$ be a graph. Vertices $u, v \in V$ are *adjacent* (or *neighbors*) if there is some edge $e \in E$ such that $\phi(e) = \{u, v\}$. Two edges $e, f \in E$ are *adjacent* if $\phi(e) \cap \phi(f) \neq \emptyset$; *i.e.*, they share a vertex. If $v \in \phi(e)$, we say that the vertex v and the edge e are *incident*. The vertex v is called *isolated* if it is not the endpoint of any edge.

The collection of edges $E' \subseteq E$ is called a set of *multiple edges* if $|E'| \geq 2$ and for all e_1, e_2 in E', $\phi(e_1) = \phi(e_2)$; that is, all of the edges in E' have the same endpoints.

Problem 5. Draw a graph G having three different sets of multiple edges, E_1, E_2, and E_3, such that $|E_1| = 2$, $|E_2| = 3$, and $|E_3| = 5$.

Problem 6. What is the greatest size a graph can have if its order is 4 and it has neither a loop nor a set of multiple edges? Draw such a graph, and give its triple.

Problem 7. What is the greatest size a graph can have if its order is 5 and it has neither a loop nor a set of multiple edges? Draw such a graph, and give its triple.

Problem 8. What is the greatest size a graph can have if its order is n ($n \geq 1$) and it has neither a loop nor a set of multiple edges? Prove your assertion. (*Hint*: you might want to use basic combinatorics here to count something...)

Definitions. A graph $G = (V, E, \phi)$ is called *simple* if it has no loops and no multiple edges. In this case, every edge has exactly two endpoints, and no edges have the same image. Essentially, this allows us to get rid of ϕ and simply think of each $e \in E$ as a pair of vertices. Thus, we often think of simple graphs as *pairs* (V, E), rather than triples. By the way, a simple graph having the greatest possible size for a given order n (as in Problem 8) is called a *complete graph of order* n, and is denoted by K_n.

Problem 9. Draw as many fundamentally different *simple* graphs as you can, each having order 4 and size 3. (Compare this problem with Problem 4).

Problem 10. Describe how you would model the structure of the internet using a graph. Is your graph a simple graph or not? Please explain.

Definitions. Let $G = (V, E, \phi)$ be a graph, and let $v \in V$ be a vertex. the *degree* of v in G, denoted $d_G(v)$, or simply $d(v)$ when G is understood, is defined by

$$d(v) = |\{e \in E \mid |\phi(e)| = 2 \text{ and } v \in \phi(e)\}| + 2|\{e \in E \mid \phi(e) = \{v\}\}|.$$

That is, we count every edge having v as an endpoint either once (if e is not a loop incident v) or twice (if e *is* a loop incident v). It's really easy to read the degree of a vertex from a pictorial representation of a graph: the degree of v will be the number over "edge ends" (often called the *germs* of edges) coming out of v.

Note. Sometimes it's more convenient to define $d_G(v)$ to be the number of edges e satisfying $v \in \phi(e)$. How does this differ from the above definition, and how are they related? When do these notions of degree coincide?

Problem 11. Give the degree of each vertex in the graph in Problem 1.

Problem 12. If G has order n and size m, what's the largest degree a vertex $v \in V(G)$ can have? What's the smallest? (*Hint*: the second question might depend on the order of the graph...)

Problem 13. Let $G = (V, E, \phi)$ be a graph. Prove the following equality:

$$\sum_{v \in V} d(v) = 2|E|.$$

Problem 14. Let G be a graph with order n and size m. What is the average degree of a vertex v in G? (*Hint*: See Problem 13.)

Definition. Let $d \geq 0$ be an integer. The graph G is called a *d-regular* graph if every vertex in G has degree d. G is simply called *regular* if there is some integer d such that G is d-regular.

Problem 15. Draw three fundamentally different 3-regular graphs. Draw two fundamentally different *infinite* 4-regular graphs.

Problem 16. Suppose G is a 2-regular simple graph of a given finite order, n. Describe what G *must* look like.