## *Summary of eigenvalues and eigenvectors*

Here is the scenario with which we're faced: we are given a succession of *states* of some kind, represented by a series of time-dependent vectors, $\{\vec{s}(t)\}_{t=0}^{\infty}$, that satisfy $\vec{s}(t+1) = T(\vec{s}(t))$ for some linear transformation $T$. That is,

$$\begin{bmatrix} s_1(t+1) \\ s_2(t+1) \\ \vdots \\ s_n(t+1) \end{bmatrix} = T \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_n(t) \end{bmatrix} = \begin{bmatrix} a_{11}s_1(t) + a_{12}s_2(t) + \cdots + a_{1n}s_n(t) \\ a_{21}s_1(t) + a_{22}s_2(t) + \cdots + a_{2n}s_n(t) \\ \vdots \\ a_{n1}s_1(t) + a_{n2}s_2(t) + \cdots a_{nn}s_n(t) \end{bmatrix},$$

for some constants $a_{ij}$. (Generally, for our purposes, those constants will be real numbers.)

As we are reminded by the final term in the above chain of equalities, a linear transformation is really little more than a matrix: if we let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

then $\vec{s}(t+1) = T(\vec{s}(t)) = A\vec{s}(t)$, where the final product is simply matrix multiplication.

What we're often trying to understand is the long-term, or *asymptotic*, behavior of the vectors $\vec{s}(t)$: what happens to $\vec{s}(t)$ as $t \to \infty$, given some *initial state* $\vec{s}(0)$?

To approach a solution to this question, we might try beginning with our initial state and applying our transformation $T$ (*i.e.*, multiplying by the matrix $A$) over and over. For instance,

$$\vec{s}(2) = T(\vec{s}(1)) = A\vec{s}(1) = AT(\vec{s}(0)) = AA\vec{s}(0) = A^2\vec{s}(0).$$

More generally, $\vec{s}(t) = A^t\vec{s}(0)$ for any $t \geq 0$.

This is where things often become difficult: raising a matrix $A$ to higher and higher powers is generally very hard to do. However, in certain circumstances our work is greatly simplified.

Given a square matrix $A$, the vector $\vec{v}$ is called an *eigenvector* for $A$, with corresponding *eigenvalue* $\lambda$, if $A\vec{v} = \lambda\vec{v}$. That is, left multiplication of $\vec{v}$ by the *matrix* $A$ can be accomplished simply by multiplication by a *number*, $\lambda$, a decidedly easier operation. Thus for an eigenvector $\vec{v}$, we see that

$$A^t\vec{v} = A^{t-1}(A\vec{v}) = A^{t-1}(\lambda\vec{v}) = \lambda A^{t-1}\vec{v} = \cdots = \lambda^t\vec{v}$$

inductively.

The next-best scenario arises when, although our vector of interest is not an eigenvector itself, we know we have a basis of eigenvectors, $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$. (Recall that a *basis* $B$ of a vector space $V$ is a collection of vectors that are linearly independent, so that no one of them can be written as a linear combination of the others, and that span the vector space: every vector in $V$ can be written as a linear combination of the vectors in $B$.) In this case, any vector $\vec{v}$ can be written $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ for the eigenvectors $\vec{v}_i$ and some constants $a_1, a_2, ...a_n$.

Now to apply $A$ to $\vec{v}$ we use this decomposition and properties of linearity of matrix multiplication:

$$\begin{aligned} A\vec{v} &= A(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots a_n\vec{v}_n) \\ &= a_1A\vec{v}_1 + a_2A\vec{v}_2 + \cdots + a_nA\vec{v}_n \\ &= a_1\lambda_1\vec{v}_1 + a_2\lambda_2\vec{v}_2 + \cdots + a_n\lambda_n\vec{v}_n. \end{aligned}$$

More generally,

$$A^t\vec{v} = a_1\lambda_1^t\vec{v}_1 + a_2\lambda_2^t\vec{v}_2 + \cdots a_n\lambda_n^t\vec{v}_n.$$

Often the most useful scenario is the one in which all but one of our eigenvalues has modulus (*i.e.*, absolute value) less than 1. Note that if $|\lambda_i| < 1$, then $\lim_{t\to\infty}\lambda_i^t = 0$. Suppose that $|\lambda_i| < 1$ for all $i \geq 2$, while $|\lambda_1| \geq 1$. (In this case we say that $\lambda_1$ is the *dominant* eigenvalue.) Returning to our formula for $A^t\vec{v}$ above, we see

$$\lim_{t\to\infty} A^t\vec{v} = a_1\lambda_1^t\vec{v}_1,$$

so in the long term our state vector $\vec{v}$ becomes parallel to the eigenvector $\vec{v}_1$ corresponding to the dominant eigenvalue $\lambda_1$.

As an example, consider the case that arose in our attempt to understand the hyperbolic graph $G(5,4)$. In this graph we identified two types of vertices; we let $o_k$ be the number of vertices of the first type in the sphere $S_k(v_0)$ (for $k \geq 1$) and $t_k$ be the number of vertices of the second type in this sphere. It was not hard to see that

$$o_{k+1} = 2o_k + t_k \quad \text{and} \quad t_{k+1} = o_k + t_k.$$

Thus if we let $\vec{s}(t) = \begin{bmatrix} o_k \\ t_k \end{bmatrix}$,

$$\vec{s}(t+1) = A\vec{s}(t), \text{ where } A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

*Mathematica* tells us that the eigenvalues of this matrix are $\frac{3}{2} \pm \frac{\sqrt{5}}{2}$. Clearly one of these, $\frac{3}{2} - \frac{\sqrt{5}}{2}$, lies on the interval $(0,1)$, while the other, $\frac{3}{2} + \frac{\sqrt{5}}{2}$, dominates. Thus the eigenvector $\begin{bmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} \\ 1 \end{bmatrix}$ corresponding to $\frac{3}{2} + \frac{\sqrt{5}}{2}$ tells us the asymptotic distribution of the vertices of each type in the sphere $S_k(v_0)$. In particular, if we normalize this eigenvector, we get the relative proportion of each type of vertex. The normalized vector we seek is

$$\begin{bmatrix} -\frac{1}{2} + \frac{\sqrt{5}}{2} \\ \frac{3}{2} - \frac{\sqrt{5}}{2} \end{bmatrix} \approx \begin{bmatrix} 0.618034 \\ 0.381966 \end{bmatrix}.$$

Thus in the limit slightly fewer than two-thirds of the vertices in $S_k(v_0)$ are of type 1.

This sort of analysis is often called *spectral* analysis, as the collection of eigenvalues of a matrix is called its *spectrum.* Recall we also undertook a spectral analysis of the "use it or lose it" trees in which we considered the probability that we occupied a certain "state" in the process of constructing such a tree.

If you have any questions about either of these applications or the open problems associated to them, or if you'd like to learn more about spectral analytic methods, please let me know.