2009 Summer REU
Groups, Graphs, and Geometry

What have we learned?

We all came here knowing simple things, like Part II of the Fundamental Theorem of Calculus: if F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(\xi)d\xi = F(b) - F(a).$$

And some of us knew what a metric was. As we *all* know now, a *metric* on a set X is a function $\rho : X \times X \to \mathbb{R}$ satisfying

1. $\rho(x, y) \geq 0$ for all $x, y \in \mathbb{R}$ and $\rho(x, y) = 0 \Leftrightarrow x = y$,
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathbb{R}$, and
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in \mathbb{R}$.

But what else have we learned?

We've learned a lot about graphs. Once we got past the easy things ("a graph G is an ordered triple (V, E, ϕ) such that yadda yadda...") we studied more weighty properties of graphs, and more interesting constructions. We defined the *independence number* $\alpha(G)$, the *chromatic polynomial* $\pi(G; k)$, the *connectivity* $\kappa(G)$, and even the *average connectivity* $\overline{\kappa}(G)$. This last one was tricky, it was given by

$$\overline{\kappa}(G) = \frac{1}{\binom{|V|}{2}} \sum_{u \neq v \in V} \kappa_G(u, v),$$

where $\kappa_G(u, v)$ is the largest number of pairwise internally disjoint (u, v)-paths we can find.

But that's not all! We also learned a lot about fractals, and the *iterated function systems* that might be used to describe them. Recall that if (X, ρ) is a metric space, then a *contraction* is any function $f : X \to X$ satisfying $\rho(f(x), f(y)) \leq r\rho(x, y)$ for all $x, y \in X$ and for some fixed $r \in (0, 1)$. An iterated function system prescribes a collection of contractions $\{f_1, ..., f_n\}$, and we would like to find a compact subset T of X such that $T = \bigcup_{i=1}^n f_i(T)$. (We call such T the *invariant set* of the IFS with the given contractions.) *Sierpiński's triangle* was one such set T.

At this point we discovered there's a bridge between graphs and fractals: *directed graphs* (or *digraphs*) can be used to describe a generalization of IFSs...and the same kind of linear algebra one can use to analyze these "digraph IFSs" can be be used to understand questions involving the average connectivity of larger and larger graphs.

Wow! Isn't it nice when math works out so well? What new wonders await us in the next seven weeks? Time only will tell!