

Graph Theory, Exercise Set 3

Working together in whatever way you find most useful, beneficial, and efficient, please try to solve as many of the following graph-theoretic exercises as you can before we meet again tomorrow. Please be prepared to present your solutions and findings to the group when we next meet.

1. Define the collection of trees $\{T_n\}_{n \geq 0}$ in the following fashion:
 - (a) T_0 is a single vertex with no edges.
 - (b) T_1 is a tree with three leaves, all connected to a single central vertex.
 - (c) T_{n+1} is formed by adding two leaves to each leaf of T_n .

Find a formula (depending on n) giving the number of vertices in T_n . (*Hint*: find a pattern from the first several terms, and use a certain summation formula.)

2. The *independence number* $\alpha(G)$ of a finite graph G is the cardinality of a largest set of vertices G that induces a null subgraph. That is,

$$\alpha(G) = \max\left\{|U| \mid U \subseteq V(G), u, v \in U \Rightarrow \{u, v\} \notin E(G)\right\}.$$

This gives yet *another* measure of the “integrity” of a graph.

- (a) For T_n given in the previous problem, describe a set $U_n \subseteq V(T_n)$ of vertices that witnesses $\alpha(T_n)$. (*Hint*: your set U_n should be very “orderly.”) Use this U_n to find a formula for $\alpha(T_n)$ in terms of n . (If you’ve chosen the right set U_n , the same kind of trick you used in Exercise 1 should work to find this formula.)
 - (b) Compute $\lim_{n \rightarrow \infty} \frac{\alpha(T_n)}{|V(T_n)|}$.
3. [**Candy bar problem**] Repeat the first two problems with T'_n constructed as before, except that all non-leaves now have degree 4 instead of degree 3.