Graph Theory, Day 3: The Geometry of Graphs

We'll start off today talking a bit about the *geometry* of graphs, in particular infinite ones.

Definition. Let G be a graph. The *graph metric* $\rho : V(G) \times V(G) \to \mathbb{R}$ on the vertex set of G is defined by letting $\rho(u, v)$ be the length of the shortest (u, v)-path in G should such a path exist, and ∞ otherwise.

Exercise 3.1. Prove that this defines a metric on $V(G)$.

Exercise 3.2. Extend the definition of ρ given above so that it defines a metric on the entire *graph* G, not just on $V(G)$. (That is, your new definition should give $\rho(x, y)$ for points x, y lying on any edge in G.)

Definition. Let $v_0V(G)$ be chosen as a *basepoint* in the graph G, and let

$$B_n(v_0) = \{v \in V(G) \mid \rho(v, v_0) \leq n\}$$

for any $n \in \mathbb{N} \cup \{0\}$. Then the *asymptotic connectivity of G relative to* v_0, is defined by

$$\kappa_a(G) = \lim_{n\to\infty} \kappa(B_n(v_0)),$$

should this limit exist.

It is easy to prove some simple results:

Exercises 3.3. Prove that if G is connected, then $\kappa_a(G) \geq 1$. Also, prove that if G has *bounded maximal degree* Δ, then $\kappa_a(G) \leq \Delta$.

Other facts are more nontrivial:

Theorem 3.4. **[Bahls]** Let $G = G(d, f)$ be as above, with G infinite and $f \geq 4$. Then $\kappa_a(G) < 2$ if and only if G is hyperbolic (that is, if and only if $\alpha(G) > 0$).