Graph Theory, Day 2: Paths, Connectivity, Et Cetera

Many of the questions we've got to ask you about graphs end up dealing with *paths*, particularly the notion of *connectivity.* Key to understanding these ideas is the following

Definitions. A *walk of length* k in a graph G is and alternating sequence $w = (v_0, e_1, v_1, ..., e_k, v_k)$ such that $v_i \in V(G)$ for $i = 0, ..., k$ and $e_i = \{v_{i-1}, v_i\} \in E(G)$ for $i = 1, ...k$. The *initial* and *terminal* vertices $\iota(w) = v_0$ and $\tau(w) = v_k$ of the walk are defined in the obvious way; w is called a (v_0, v_k)-*walk.*

A walk is called a *trail* if $i \neq j \Rightarrow e_i \neq e_j$, and a *path* if $i \neq j \Rightarrow v_i \neq v_j$. If the walk w satisfies $v_0 = v_k$, $k \geq 1$, it is called *closed*; a closed trail is called a *circuit*, and a closed walk in which only v_0 and v_k are equal is called a *cycle.*

Try your hand at proving the following

Theorem 2.1. If there is a (u, v)-walk in G, then there is a path in G from u to v.

Knowing what paths are allows us to define what it means for G to be *connected*, and what G's *connected components* are. With the correct definition, you should be able to prove the following

Theorem 2.2. Define the relation $\sim$ on $V \times V$ by

$$u \sim v \Leftrightarrow \exists \text{ a } (u, v)-\text{path in } G.$$

Then $\sim$ is an equivalence relation on V. (What are its *equivalence classes*?)

Definition. We say a connected graph G is a *tree* if given $u \neq v$ in $V(G)$, there is a *unique* (u, v)-path in G. A *forest* is a union of trees.

Theorem 2.3. A tree with n edges has __________ vertices.

Definitions. Let G be any (locally finite) graph, $|V(G)| \geq 2$, and let $u \neq v \in V(G)$. Then the *connectivity of* u *and* v, *relative to* G, denoted $\kappa_G(u, v)$, is the cardinality of the largest collection of mutually disjoint (u, v)-paths. (Here, clearly, we allow paths that share the common endpoints u and v.) The *connectivity* of G, denoted $\kappa(G)$, is the minimum of $\kappa_G(u, v)$ over all ordered pairs $(u, v) \in V \times V$, $u \neq v$.

Exercises 2.4. What can you say if $\kappa(G) = 0$? What if $\kappa(G) = 1$? What is $\kappa(T)$ for a tree T? What is $\kappa(K_n)$ for the complete graph K_n, $n \geq 2$? What about for the circuit graph, C_n, $n \geq 3$?

Definition. Let G be finite. The *total connectivity* of G, denoted $\tau(G)$, is defined by

$$\tau(G) = \sum_{\{u,v\} \subseteq V, u \neq v} \kappa_G(u, v).$$

The *average connectivity* of G, denoted $\bar{\kappa}(G)$, is defined by

$$\frac{1}{\binom{|V|}{2}} \sum_{\{u,v\} \subseteq V, u \neq v} \kappa_G(u, v) = \frac{1}{\binom{|V|}{2}} \tau(G).$$

Exercises 2.5. Compute $\bar{\kappa}(G)$ for each of the following cases:

1. G is a tree,
2. $G = K_n$, $n \geq 2$, and
3. G is a "patch" inside of the infinite graph $G(4, 4) = \mathbb{Z} \times \mathbb{Z}$.

Related to connectivity are other measures of a graph's "integrity."