*Graph Theory, Day 1: The Basics*

The first order of business is to establish some ground rules regarding *graphs*, one of the most focal points of this summer's study.

Even more fundamentally, we will assume knowledge of several important underlying mathematical concepts. Let's take a moment to make sure that we're all familiar with the following notions:

- set, empty (null) set, subset, union, intersection
- distinct set, disjoint set, set difference, symmetric difference, powerset
- function, one-to-one function (injection), onto function (surjection), bijection
- cardinality, finite set, countable set, uncountable set
- ordered pair, ordered tuple, Cartesian product, binary relation, equivalence relation
- pigeonhole principle, principle of mathematical induction, well-ordering principle
- naturals, integers, rationals, reals, complex numbers

Whew! We're now ready to tackle graphs.

**Definition.** A *graph* (specifically, an *undirected* graph) $G$ is an ordered triple $(V, E, \phi)$ such that

1. $V \neq \emptyset$,
2. $V$ and $E$ are disjoint, and
3. $\phi : E \to \mathcal{P}(V)$, the powerset of $V$, such that $|\phi(e)| \in \{1, 2\}$ for all $e \in E$.

Often we don't even mention $\phi$ explicitly, if we know what it is we're talking about. Slight refinements and modifications allow us to define simple graphs and directed graphs: $G$ is called *simple* if additionally

3.a. $|\phi(e)| = 2$ for all $e \in E$, and

3.b. $\phi$ is injective.

A 4-tuple $G = (V, E, \iota, \tau)$ is called a *directed graph* if conditions (1) and (2) are still met for $V$ and $E$, and additionally $(3')$ $\iota, \tau : E \to V$ for all $e \in E$. We can make this $G$ *simple* by demanding that $(3'.\text{a})$ $\iota(e) \neq \tau(e)$ for all $e \in E$, and that $(3'.\text{b})$ $e_1 \neq e_2 \Rightarrow \iota(e_1) \neq \iota(e_2)$ or $\tau(e_1) \neq \tau(e_2)$.

Before we get any further, you should understand what is meant by a *geometric realization* geom($G$) of a graph $G$. Very often we will conveniently forget that $G$ is an abstract combinatorial object and work only with an explicit geometric realization of it, identifying the two in our minds.

Here are a number of important concepts concerning graphs that we ought to be familiar with:

- adjacent vertices or edges, incident edges, loops, parallel edges
- degree, in-degree, out-degree, isolated vertex
- null graph, regular graph, path graph, cycle graph, bipartite graph, complete graph
- finite graph, infinite graph, locally finite graph

*Subgraphs* $G'(V', E', \phi')$ of $G = (V, E, \phi)$ are defined by restriction, essentially: all we demand is that $V' \subseteq V$, $E' \subseteq E$, and that $\phi'$ agrees with $\phi$ wherever it's defined. It's easy to define the subgraphs $G(V')$ or $G(E')$ of $G$ *induced* by vertex sets $V'$ or by edge sets $E'$. (What ought they be?)

Below are some results important enough to single them out for our attention. Let's see for how many of these we can come up with proofs.

**Proposition 1.1.** Let $n \in \mathbb{N}$. The number of edges in the complete graph $K_n$ on $n$ vertices is $\binom{n}{2} = \frac{n(n-1)}{2}$.

**Proposition 1.2.** Let $m, n \in \mathbb{N}$. The number of edges in the complete bipartite graph $K_{m,n}$ is $mn$.

**Theorem 1.3.** For any graph $G = (V, E, \phi)$ we have

$$\sum_{v \in V} d_G(v) = 2|E|.$$

**Corollary 1.4.** In any graph $G$, the number of vertices of odd degree is even.

**Proposition 1.5.** Not every graph is *planar*, that is, not every graph has a geometric realization that can be drawn in the plane without drawing edges that cross elsewhere than vertices.

**Proposition 1.6.** The graph $K_4$ is planar, but $K_n$ is not, for $n \geq 5$. However, $K_5$ has genus 1, meaning it can be drawn on the surface of a *torus* with a single handle, without crossing edges elsewhere than vertices. What about $K_6$? (This exercise introduces us to *topological* graph theory!) Note: there is a much more general result concerning the number of handles in a torus on which a complete graph can be drawn. Let the *genus* $\gamma(G)$ of the graph $G$ be the minimum number of handles in a torus on which $G$ can be drawn without crossing edges. Then we have

**Theorem 1.7. [Mayer-Ringel-Youngs]** For any $n \geq 3$, $\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$.