Geometry, Day 3: Metrics

In order to get a handle on some of the graph theory I find interesting, we'll need to understand what a *metric space* is.

Definition. Let X be a set. We call a function $\rho : X \times X \to \mathbb{R} \cup \{\infty\}$ a *metric* on X if the following properties hold:

1. (positive definiteness) $\rho(x, y) \geq 0$ for all $x, y \in X$; moreover $\rho(x, y) = 0 \Leftrightarrow x = y$,
2. (symmetry) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$, and
3. (triangle inequality) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$.

Together, the pair (X, ρ) is called a *metric space.*

Metrics essentially make precise the notion of the "distance" between two points in a space.

You should be able to prove each of the following results:

Proposition 3.1. Any set X can be given the *discrete metric*, defined by $\rho(x, y) = 1$ if $x \neq y$ and $\rho(x, x) = 0$.

Proposition 3.2. The set $\mathbb{R}$ of real numbers admits the metric ρ defined by $\rho(x, y) = |x - y|$.

Proposition 3.3. The set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ admits the *standard Euclidean metric* ρ_2 defined by $\rho_2\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Proposition 3.4. The set $\mathbb{R}^2$ also admits metrics ρ_1 and ρ_∞ defined by

$$\rho_1\big((x_1, y_1), (x_2, y_2)\big) = |x_2 - x_1| + |y_2 - y_1|$$

and

$$\rho_\infty\big((x_1, y_1), (x_2, y_2)\big) = \max\{|x_2 - x_1|, |y_2 - y_1|\}.$$

Definition. Let x and y be points in the metric space M. If p is a path from x to y witnessing $\rho(x, y)$, we call p an (x, y)-*geodesic.*

Exercise 3.6. Is the *inverse* of a geodesic always a geodesic? Is the *concatenation* of geodesics always a geodesic?

Definitions. A *geodesic triangle* $\Delta(x, y, z)$ in a metric space M is the union of any three geodesics $p(x, y)$, $p(y, z)$, and $p(z, x)$. Given a number $\delta \geq 0$, the triangle Δ is called δ-*thin* if for every point a lying on one of the legs of Δ, there is a point b lying in the union of the other two legs such that $\rho(a, b) \leq \delta$. If there is some fixed $\delta \geq 0$ such that every geodesic triangle in M is δ-thin, we call M a δ-*hyperbolic* metric space. M is called simply *hyperbolic* if here is some δ for which it is δ-hyperbolic.

Exercises 3.7. Is the Euclidean plane $\mathbb{R}^2$ δ-hyperbolic for some δ? What about any *finite* graph G? What about the integer lattice graph $\mathbb{Z} \times \mathbb{Z}$? What about an infinite tree? (These questions require knowledge of the path metric in a graph, given in the Graph Theory seminar notes from Day 3.)