# Attempted and Successful Orderings: Braid and Baumslag-Solitar Group

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# Beginnings

**Assumption.** I will assume that Magnus' transformation has been covered, and that everybody is adequetely primed in this regard.

**Braid Group Primer** The braid group on 3 strings has a normal form, such that every element can be brought into the form $\Delta^i p$. Where, $i \in \mathbb{Z}$ and $\Delta = aba = bab$, in the three string case. And p is a positive word such that $\Delta = aba = bab$ is not a subword of p.

# Later Beginnings

When we apply Magnus' method to $B_3$ we get that:

$$\begin{aligned}(1+a)(1+b)(1+a) &= 1+2a+a^2+b+ab+ba+aba \\ &= 1+2b+b^2+a+ab+ba+bab \\ &= (1+b)(1+a)(1+b)\end{aligned}$$

And by cancelling like terms, we get $b^2 = a^2 + a - b$. And all is still well.

Note, that the properties we have so far imply an algebra normal form like this:

$$\sum_{i=0}^{\infty} \Delta^i \left( \sum_{j=0}^{\infty} \alpha_{j_i} a^j + \beta_i b + \sum_{k=1}^{\infty} \gamma_{k_i} a^k b + \sum_{l=0}^{\infty} \delta_{l_i} a^l b + \sum_{m=2}^{\infty} \epsilon_{m_i} b a^m b \right)$$

# New Relations

But, since the order should be irrelevant when applying $b^2 = a^2 + a - b$ to $b^3$ , we must also have that

$$\begin{aligned} b^3 = b^2 b = (a^2 + a - b)b &= a^2 b + ab - b^2 \\ &= ba^2 + ba - b^2 = b(a^2 + a - b) = bb^2 = b^3 \end{aligned}$$

But, again, cancelling like terms implies that $ba^2 = a^2 b + ab - ba$.

Now our normal form should look something like this:

$$\sum_{i=0}^{\infty} \Delta^i \left( \sum_{j=0}^{\infty} \alpha_{j_i} a^j + \beta_i b + \gamma_i ba + \sum_{l=0}^{\infty} \delta_{l_i} a^l b \right)$$

# Even more relations

But wait, there's more. Notice that, $\Delta a = (bab)a = b(aba) = b(bab) = b^2ab$. This implies that in the algebra:

$$\Delta a = b^2ab = (a^2 + a - b)ab = a^3b + a^2b - \Delta$$

Rearranging this we get that:

$$a^3b = \Delta a + \Delta - a^2b$$

New normal form:

$$\sum_{i=0}^{\infty} \Delta^i (\sum_{j=0}^{\infty} \alpha_{j_i} a^j + \beta_i b + \gamma_i a^2 b + \delta_i ab + \epsilon_i ba)$$

# Potential problem:

Since, in the group $a^{-1} = b^{-1}a^{-1}b^{-1}ab$, we should have that:

$$(1 - a + a^2 - a^3 + \cdots) =$$

$$(1-b+b^2-b^3+\cdots)(1-a+a^2-a^3+\cdots)(1-b+b^2-b^3+\cdots)(1+a)(1+b)$$

in the algebra. But it certainly doesn't look like it.

# No problem at all

When, in the algebra, we "mod" out by $\langle b^2 = a^2 + a - b \rangle$ we really should be "moding" out by every consequence of $(1+a)(1+b)(1+a) = (1+b)(1+a)(1+b)$. One of which is the problem on the previous slide.

And so, we need to start adding more relations, presumably relations that handle formal inverses. And so, let's start with some trivial properties we will want.

$$\begin{aligned} ab(1+a)^{-1} &= ab(1 - a + a^2 - a^3 + a^4 - \cdots) \\ &= (ab - aba + aba^2 - aba^3 + aba^4 - \cdots) \\ &= (ab - \Delta + \Delta a - \Delta a^2 + \cdots) \\ &= (ab - \Delta + b\Delta - b^2\Delta + \cdots) \\ &= (ab - bab + b^2ab - b^3ab + \cdots) = (1+b)^{-1}ab \end{aligned}$$

Similarly $ba(1+b)^{-1} = (1+a)^{-1}ba$

# Even more (slightly less obvious)

Since $b^{-1}ab = aba^{-1}$ in the group, we must have that:

$$\begin{aligned}
(1+b)^{-1}(1+a)(1+b) &= (1+a)(1+b)(1+a)^{-1} \\
\Rightarrow (1+b)^{-1}(1)(1+b) + (1+b)^{-1}a(1+b) &= (1+a)(1)(1+a)^{-1} + (1+a)b(1+a)^{-1} \\
\Rightarrow 1 + (1+b)^{-1}(a+ab) &= 1 + (b+ab)(1+a)^{-1} \\
\Rightarrow (1+b)^{-1}(a+ab) &= (b+ab)(1+a)^{-1} \\
\Rightarrow (1+b)^{-1}a + (1+b)^{-1}ab &= b(1+a)^{-1} + (ab)(1+a)^{-1} \\
\Rightarrow (1+b)^{-1}a + (1 - bab + b^2ab - \cdots) &= b(1+a)^{-1} + (ab + aba - aba^2 + \cdots)^{-1} \\
\Rightarrow (1+b)^{-1}a + (1 - \Delta + b\Delta - \cdots) &= b(1+a)^{-1} + (ab + \Delta - \Delta a + \cdots) \\
\Rightarrow (1+b)^{-1}a &= b(1+a)^{-1}
\end{aligned}$$

In similar ways we get that

$$(1+a)^{-1}b = a(1+b)^{-1}$$

and

$$(1+a)^{-1}(1+b)^{-1}a = b(1+a)^{-1}(1+b)^{-1}$$

# $\mu(\Delta^{-1})$

you with tedious calculations that would take up 12 slides (I know this number because I was originally going to bore you with tedious calculations that took up 12 slides), suffice it say that
Now, we a chance of finding out what $\mu(\Delta^{-1})$ looks like. To avoid boring

$$\mu(\Delta^{-1}) = (1+b)^{-1} - b(1+a)^{-2} - (1+a)^{-2}b + \Delta(b(1+a)^{-2})$$

But, looking at those ”things” that will contribute to the coefficient on ab. We derive that

$$ab - 2ab + 3ab - 4ab + 5ab - \cdots$$

which doesn’t diverge uniformily. This is a problem

## Possible solution

The problem of using Magnus' method for the braid group, $B_3$, seems to come from the formal inverses. What if, instead of mapping into the "monoid algebra(ring)" we mapped into the group algebra(ring), forcing the cancellation of inverse pairs the same way we forced the braiding relation.

(Note: for simplicity I will now have $B = b^{-1}$ and $A = a^{-1}$)

That is instead of mapping $\mu : B \longmapsto (1 - b + b^2 - b^3 + \cdots)$, we map $\mu : B \longmapsto (1 + B)$ and require

$$\begin{aligned} 1 &= (1+b)(1+B) = 1 + b + B + bB = 1 + b + B \\ 1 &= (1+a)(1+A) = 1 + a + A + aA = 1 + a + A \\ \Rightarrow B &= -b \\ \Rightarrow A &= -a \end{aligned}$$

Note: these relations also require that $a^2 = -Aa = 0 = -Bb = b^2$

# Terminal Problem for the Braid Group

In the braid group, these new relations, combined with those derived from the braiding relation imply:

$$0 = b^2 = a^2 + a - b = a - b$$

or, $a = b$. Which is a problem. This implies something in the group that isn't true. (Similar results hold for other two generator large-type Artin groups)

# Baumslag-Solitar groups

If we apply this new method to BS(1,n), where $n \geq 3$ we get more useful results. (Note: that we still have that $A = -a$, $B = -b$, and $a^2 = -Aa = 0 = -Bb = b^2$)

And so, forcing $ab = ba^n$ to be true in the algebra, we get that:

$$\begin{aligned}(1+a)(1+b) = (1+b)(1+a)^n = (1+b)(1+na)\\ \Rightarrow 1+a+b+ab = 1+na+b+n(ba)\\ \Rightarrow ba = \frac{1-n}{n}a\end{aligned}$$

And also, in the algebra $ab = b^n a = 0a = 0$

# What an element maps to

$$
\begin{aligned}
\mu(b^m a^l B^k) &= (1+b)^m(1+a)^l(1+B)^k \\
&= (1+mb)(1+la)(1-kb) \\
&= 1+mb+la+(ml)ba-kb-(mk)b^2-(lk)ab+(mlk)aba \\
&= 1+mb+la+(ml)ba-kb \\
&= 1+(m-k)b+l(1-\frac{m(n-1)}{n})a
\end{aligned} \tag{1}
$$

Now for an arbitrary element to map to the identity we must have that$m-k=0$, or m=k. Also, if $1-\frac{m(n-1)}{n}=0$, this whould require $m=\frac{n}{n-1}$. But, if $n \geq 3$, then $\{\frac{n}{n-1}\} \nsubseteq \mathbb{Z}$. And so, $l=0$. Therefore $b^m a^l B^k = b^m a^0 b^{-m} = 1$, as desired. And so, $\mu$ is injective.

# The product of two elements in the algebra

$$\mu(b^m a^l B^k)\mu(b^{m'} a^{l'} B^{k'})$$

$$= (1 + (m-k)b + l(1 - \frac{m(n-1)}{n})a)(1 + (m'-k')b + l'(1 - \frac{m'(n-1)}{n})a)$$

$$= 1 + [(m-k)+(m'-k')]b + [l(1-\frac{m(n-1)}{n}) + l'(1-\frac{m'(n-1)}{n}) + (m-k)l'(1-\frac{m'(n-1)}{n})(1-\frac{m'(n-1)}{n})]a$$

With this we get the one of the non-obvious ordering property that we are required.