Random Trees

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Some Quick Definitions

Tree: A tree is an acyclic, undirected graph.

Leaf: A leaf is a vertex that has degree 1.

We will be talking about different random process we have been working with to produce trees.

Classical Random Trees

Construction: Classical Random Trees are constructed by starting with a single vertex v_0 and at each iteration, adding a new vertex that has a uniform probability of connecting to an already existing vertex v_i by an edge.

Let $E(L_n)$ be the expected number of leaves that each tree will have, where n is the number of vertices. Let $E(\%_n) = \frac{E(L_n)}{n}$.

Probabilities of Classical Random Trees

We start with a single vertex, and at each step, t, pick a random vertex in which to grow the new vertex. We want to look at the probability that $v_1, v_2, \ldots, v_n$ are leaves.

The probability that the n^{th} vertex added will be a leaf, denoted $P(n)$, is 1.

$$P(n-1) = \frac{n-2}{n-1}$$

$$P(n-2) = (\frac{n-3}{n-2})(\frac{n-2}{n-1}) = \frac{n-3}{n-1}$$

$$\vdots$$

$$P(n-i) = \frac{n-(i+1)}{n-1}, \text{ where } i \neq (n-1)$$

As we continue to follow this process we found $P(1) = \frac{1}{n-1}$.

Probabilities of Classical Random Trees (cont.)

By summing the probabilities of each vertex being a leaf, we arrived at this equation:

$$\frac{1}{n-1} + \frac{1}{n-1}\sum_{i=1}^{n-1} i = \frac{1}{n-1} + \frac{n}{2}.$$

Thus,

$$\lim_{n\to\infty} E(L_n) = \lim_{n\to\infty} \frac{1}{n-1} + \frac{n}{2} = \frac{n}{2},$$

and

$$\lim_{n\to\infty} \frac{E(L_n)}{n} = \frac{1}{2}.$$

Degree Matters

Construction: Degree Matters Trees are constructed by starting with a single vertex v_0 and we add new vertices randomly based on the degree of each vertex. I.e. The higher the degree of a vertex, the more likely it is to obtain a new vertex and edge.

By following a similar proof as the classical case, we were able to prove that $E(\%_n)$ converges to $\frac{2}{3}$.

Use it or Lose it Trees

Construction: Use it or Lose it Trees are constructed by establishing initial values m and n, where m is a counter value that dictates a vertex's ability to be attached to new vertices, and n is the number of new vertices that are created at each iteration.

Two things happen at each iteration:

- New vertices are added
- Counter values are altered

States and Transition Graphs

We have also defined a method for labeling what "state" the trees are in at a specific time. We let $S = (x_1, x_2, \ldots, x_m)$ where x_i is the number of vertices that have counter value $= i$.

Transition Graphs are graphs that show which states can be reached from every state. When looking at these states for different initial conditions, we noticed that some states are transient, and others are recurrent.

Probabilities of Use it or Lose it Trees

Using the transition graphs, we were able to build probability matrices for the recurrent states, to grasp P(vertex is a leaf). The probability matrix for the recurrent states of $m = 2, n = 1$ is as follows.

$$\begin{array}{ccc} & (1,2) & (2,2) \\ (1,2) & \frac{2}{3} & \frac{1}{2} \\ (2,2) & \frac{1}{3} & \frac{1}{2} \end{array}$$

By finding the dominant eigenvalue, we find its corresponding eigenvector and normalize it to get this:

$$\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$$

Then by taking every recurrent state, and finding out the probability that a vertex will remain a leaf, we arrive at $P(\text{vertex is a leaf}) = \frac{1}{2}$

To be continued...

In considering the Degree matters trees, we now want to

- look at more ways to weight the importance of degree,
- and find more $E(L_n)$ and $E(\%_n)$.

To be continued...(continued)

In considering the Use it or Lose it Trees, we want to

- establish a good way to find recurrent states,
- generalize findings for more values of n
- and find out more about the eigenvectors.