

Asymptotic Connectivity and Hyperbolic Graphs

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Initial Assumptions and Definitions

- We consider only graphs G which are simple, undirected, planar, connected, one-ended, with bounded maximal degree and face length.
- Throughout, $V(G)$ and $E(G)$ refer to the vertices and edges of G , respectively.
- **Connectivity** $\kappa_G(u, v)$ of two vertices $u, v \in V(G)$:
The cardinality of a largest set of disjoint paths between u and v . Disjoint paths only have endpoint vertices in common.
- **Average Connectivity** $\bar{\kappa}(G)$ of a finite graph G :

$$\bar{\kappa}(G) = \frac{1}{\binom{|V(G)|}{2}} \sum_{u \neq v} \kappa_G(u, v)$$

Asymptotic Connectivity

Asymptotic Connectivity $\kappa_a(G, v_0)$ of an infinite graph G relative to a basepoint v_0 : Let v_0 be any vertex of G and let $B_n(v_0)$ be the *ball of radius n* centered at v_0 .

$$\kappa_a(G) = \lim_{n \rightarrow \infty} \bar{\kappa}(B_n(v_0))$$

Weak Regularity and Polynomial Growth

A graph G is *weakly regular* if there exists a well-defined **signature function** $f : \mathbb{N} \rightarrow [0, 1]$ with finite support such that:

$$f(i) = \lim_{n \rightarrow \infty} \frac{|\{v \in B_n(v_0) \mid d(v) = i\}|}{|B_n(v_0)|}$$

For infinite, weakly regular graphs G with polynomial growth and finite degree sequence $d_1 < d_2 < \dots < d_k$, Bahls has shown that $\kappa_a(G, v_0)$ exists independent of basepoint and

$$\kappa_a(G, v_0) = \sum_{i=1}^k d_i (f(d_i)^2 + 2 \sum_{j>i} f(d_i) f(d_j))$$

Hyperbolic Graphs - Growth of G and the boundary of $B_n(v_0)$

- Polynomial growth (Euclidean) - 'Area' of $B_n(v_0)$ in a graph G dominates over the boundary (or the n -sphere $S_n(v_0)$). Vertices closer to the boundary have lower connectivity, but these vanish in the limit for graphs with polynomial growth.
- Exponential growth (Hyperbolic) - 'Area' of $B_n(v_0)$ in a graph G is roughly the same size as $S_n(v_0)$. Those vertices close to the boundary with lower connectivity can no longer be ignored when computing asymptotic connectivity.

Asymptotic Connectivity in Hyperbolic Graphs

- The effect of boundary vertices is so significant that Bahls conjectures $\kappa_a(G, v_0) \leq 2$ for hyperbolic graphs G with a negligible proportion of triangular faces and degree 3 vertices. He has proved this for the *d-regular* tilings $G(d, f)$ of the hyperbolic plane.
- The subject of our research has been to generalize this bound to other hyperbolic graphs.

Modifications

For G a hyperbolic graph and $u \in B_n(v_0)$, define the *effective degree of u relative to $B_n(v_0)$* , $d_n(u)$ as

$$d_n(u) = \max\{\kappa_{B_n(v_0)}(u, x) \mid x \in B_n(v_0)\}$$

$d_n(u)$ gives the number of disjoint paths one can find leaving the vicinity of u .

If we substitute $d_n(u)$ for $d(u)$ in the signature function for G and in Bahls's formula for asymptotic connectivity, we arrive at the correct formula for $\kappa_a(G, v_0)$.

Assumptions and Vertex Types

In computing asymptotic connectivity for hyperbolic graphs G , we further assume:

- The facelength $f \geq 4$ for all faces of G
- The degree $d(v) \geq 4$ for all $v \in V(G)$.

This allows us to extend a feature of the d -regular graphs $G(d, f)$ noted by Bahls: Suppose $u \in S_n(v_0)$. Then u is one of two types of vertices:

- Type one vertices are adjacent to only one vertex in $B_n(v_0)$.
- Type two vertices are adjacent to two vertices in $B_n(v_0)$, at most one of which is in $S_n(v_0)$.

An extension of $G(d, 4)$

Consider the extension of $G(d, 4)$ by the following construction process:

1. Choose an initial vertex v_0 and assign it degree f .
2. Assign degree $d_o \geq 4$ to type one vertices and degree $d_t \geq 4$ to type two vertices. $d_o = 4$ and $d_t = 4$ are not simultaneously true.
3. Make all faces quadrilaterals.

Denote the resulting graph by $G(4, f, d_o, d_t)$. Represent the number of type one and type two vertices in $S_n(v_0)$ by o_n and t_n , respectively.

Bounding Asymptotic Connectivity of $G(4, f, d_o, d_t)$

Theorem: $\kappa_a(G, v_0)$ for $G(4, f, d_o, d_t)$ is bounded above by two.

Proof

- Derive explicit formulas for o_n , t_n , and $B_n(v_0)$ from recursive relations
- Compute $f(1) = \lim_{n \rightarrow \infty} \frac{o_n}{B_n(v_0)}$ and $f(2) = \lim_{n \rightarrow \infty} \frac{t_n}{B_n(v_0)}$
- Let $k = \max\{d_o, d_t\}$. Assume $f(k) = 1 - (f(1) + f(2))$. Input these signature function values into the modified formula for connectivity: this gives an upper bound on $\kappa_a(G(4, f, d_o, d_t), v_0)$.
- For $d_o \geq 6$ or $d_t \geq 11$, the upper bound is less than 2, leaving a finite number of cases to check more carefully. These also turn out under 2.

Trails Left Hanging - Directions for Future Research

Independence of Basepoint This is top of the list at the moment. Even for the graphs $G(4, f, d_o, d_t)$ we compute $\kappa_a(G, v_0)$ without verifying that it is a statistic independent of the choice of v_0 .

Combinatorial Curvature and Isoperimetric Number Combinatorial curvature (computed for individual vertices) and the isoperimetric number (a statistic for infinite graphs) are both measures of hyperbolicity in a graph G . The hope is to gain estimates on asymptotic connectivity from such measures and ultimately indicate their relationship to one another.