Ordering BS(1,3) Using the Magnus Transformation

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Definitions

In this presentation we will put an order on the Baumslag-Solitar group represented by

$$BS(1,3) = \langle a, b \mid ab = ba^3 \rangle$$

with normal form $b^i a^j b^{-k}$.

We will need to consider the monoid, $M = BS^+(1,3)$, which is a subset of $BS(1,3)$ generated by the noninverse elements of $BS(1,3)$, and the Magnus transformation, μ, where

$$\mu(a) = 1 + a$$

$$\mu(a^{-1}) = 1 - a + a^2 - a^3 + ...$$

Example:

$$\mu(b^2ab^{-1}) = (1+b)^2(1+a)(1-b+b^2-b^3+...)$$

Preserving Group Structure Under Magnus

$$\mu : BS(1,3) \to A_{\mathbb{Q}}(M)$$

In order for this algebra to accurately represent our group we must preserve our original relation by mapping it into the algebra as well:

$$\begin{array}{rcll} (1+a)(1+b) & = & (1+b)(1+a)^3 & \to \\ 1+a+b+ab & = & 1+3a+3a^2+a^3+b+3ba+3ba^2+ba^3 & \to \\ \cancel{1}+\cancel{a}+\cancel{b}+\cancel{a}b & = & \cancel{1}+\cancel{3}2a+3a^2+a^3+\cancel{b}+3ba+3ba^2+\cancel{b}a^3 & \end{array}$$

This leaves us with the relation

$$ba^2 = -\frac{2}{3}a - a^2 - \frac{1}{3}a^3 - ba.$$

Let $\mathcal{A} = A_{\mathbb{Q}}(M)/I$, where I is the ideal generated by the above relation.

Normality

Using the relation from the previous slide we can see that every element in the monoid algebra can be written of the form,

$$1 + \sum_{i=1}^{\infty} \beta_i b^i + \sum_{j=1}^{\infty} \alpha_j a^j + \sum_{k=1}^{\infty} \gamma_k b^k a$$

Example:

$$\begin{aligned}
\mu(ba^2) &= (1+b)(1+a)^2 \\
&= 1 + b + 2ba + ba^2 + 2a + a^2 \\
&= 1 + b + 2ba + (-\tfrac{1}{3}a^3 - a^2 - \tfrac{2}{3}a - ba) + 2a + a^2 \\
&= 1 - \tfrac{1}{3}a^3 + \tfrac{4}{3}a + ba + b
\end{aligned}$$

We must now check that μ is still injective.

Injectivity

Let $g = b^m a^l b^{-k}$

1) $m \neq k, m, k \neq 0$ and $l = 0$ $\quad \mu(b^{m-k}) = (1 + (m-k)b + \ldots)$

2) $k = 0$, and $l, m \neq 0$ $\quad \mu(b^m a^l) = (1 + mb + \ldots)(1 + la + \ldots)$

3) $m = 0$, and $l, k \neq 0$ $\quad \mu(a^l b^{-k}) = (1 + la + \ldots)(1 - kb + \ldots)$

4) $m = k = 0$ and $l \neq 0$ $\quad \mu(a^l) = (1 + la + \ldots)$

5) $m, k, l \neq 0$

For $y \in \mathcal{A}$ we will use the notation $c(y, m)$ to denote the coefficient of the monoid element m in y.

Ex: If $g = ba^2$, $\mu(g) = 1 + b + 2ba + ba^2 + 2a + a^2$, and thus $c(\mu(g), ba) = 2$.

a) $l > 0$

$$c(\mu(b^m a^l b^{-k}), a) = 1 + l - 3^{(k-m)}$$

b) $l < 0$

$$c(\mu(b^m a^l b^{-k}), a) = -l + (3^{(k-m)} - 1) \sum_{j=1}^{\infty} \binom{j+l-1}{l-1}$$

Examining the Algebra

If $c(y, b) = 0$, then we can show that $c(y, b^z a) = 0$, where $z \geq 1$ using the following theorem:

Theorem 1 *For positive integers* i, j, z, *then* $c(b^i a^j, b^z a) = (-1)^{i+1}$ *when* $i = z$ *and zero otherwise.*

Consider the terms in $y = (1+b)^m(1+a)^l(1-b+b^2-b^3\ldots)^k$ that look like $b^h a^j b^i$ where $i + h = z$. Then we have

$$c(y, b^z a) = \sum_{i=0}^{z} \binom{i+k-1}{k-1}\binom{m}{z-i}(-1)^i = \frac{\Gamma(1-k+m)}{\Gamma(1-k+m-z)\Gamma(1+z)}$$

which equals zero when $m = k$.

Ordering

Define H as the image of the group in the Algebra.

Define a strictly positive cone C on a group H.

Let $x = \sum_{i=1}^{\infty} \beta_i b^i + \sum_{j=1}^{\infty} \alpha_j a^j + \sum_{h=1}^{\infty} \gamma_h b^h a \in \mathcal{A}$.

- If $c(x, b) \neq 0$, then we will define $\tau(x) = b$.
- If $c(x, b) = 0$ then we define $\tau(x) = a$.

For all $h \in H$, let $\lambda(h) = c(h, \tau(h))$

Let $C = \{1 + x \in H \mid \lambda(x) > 0\}$

(C1) $C \cdot C \subseteq C$

(C2) $hCh^{-1} \subseteq C$, for all $h \in H$

(C3) $C \cap C^{-1} = \emptyset$, and

(C4) $C \cup C^{-1} \cup \{1\} = H$

If these hold true for C, then $h_1 < h_2 \in H \Leftrightarrow \mu({h_2}^{-1} h_1) \in C$.

(C2) $hCh^{-1} \subseteq C$, for all $h \in H$

- Consider $1 + x \in C, 1 + y \in H$
- First, assume $c(1 + x, b) = 0$.Then $1 + x = (1 + a)^l$. We also know that $1 + y = (1 + b)^m(1 + a)^l(1 + b)^{-k}$.
- Thus$(1 + y)(1 + x)(1 + y)^{-1}$ can be rewritten as $(1 + b)^i(1 + a)^j(1 + b)^{-k}(1 + a)^l(1 + b)^k(1 + a)^{-j}(1 + b)^{-i}$ $= \mu(b^ia^jb^{-k}a^lb^ka^{-j}b^{-i})$ $= \mu(b^ia^{3^kl}b^{-i})$.
- We know that $c(\mu(b^ia^{3^kl}b^{-i}), a) = 3^kl > 0$. Thus, $(1 + y)(1 + x)(1 + y)^{-1} \in C$.

We have now shown that our set C satisfies all four properties and have successfully defined an order on H.