On the Structure of A_k and $A_k(1)$

Ryan Causey
Mississippi State University
rmc107@msstate.edu

Basic Definitions

For a ring k with unity 1_k, with the set of units $U(k)$, A_k, the set of *arithmetical functions*,is the set $A_k = \{f : N \rightarrow k \mid f(1) \in U(k)\}$.

$A_k(1)$ is the set of *unity functions*, $A_k(1) = \{f \in A_k \mid f(1) = 1_k\}$.

We define the *Dirichlet convolution* $* : A_k \times A_k \rightarrow A_k$ by
$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d})$.

To evaluate $(f_1 * f_2 * ... * f_m)(n)$, we evaluate the sum

$$\sum_{D} g(d_1)g(d_2)...g(d_m)$$

taken over the set of all ordered m-tuples such that $\prod_{i=1}^{m} d_i = n$.

$< A_k, * >$ forms a group, and it is abelian if and only if k is a commutative ring.

Integer analogues

We can associate the element $m_k = \sum_{i=1}^{m} 1_k \in k$ with the positive integer $m \in N$. For much of the work we do, we are concerned with whether or not m_k permits cancelation. That is, is m_k zero or a divisor of zero.

Torsion Elements

A torsion element is an element of finite order. Obviously the identity is always a torsion element.

To search for other torsion elements in A_k, we begin by looking in $A_k(1)$.

Theorem: Suppose k is a ring with characteristic q, and fix $m \in N$. If $m \cdot a = 0 \Rightarrow a = 0$ for all $a \in k$, then A_k has no elements of order m.

Corollary: If k has characteristic zero and, for any $m \in N$, $1_k + 1_k + \ldots + 1_k$ for m summands is not a divisor of zero, then $A_k(1)$ is torsion free. If k has characteristic q, then $A_k(1)$ is has no element of order m if $gcd(m, q) = 1$.

A few results

Theorem: For a ring k with prime characteristic q, A_k has an element of order q if and only if $a^q = 0$ has a nontrivial solution in the ring k.

Proof of the theorem hinges on partitioning the ordered q-tuples. In the commutative case, we look at equivalence classes consisting of permutations of the q-tuples. In the noncommutative case, we look at cancelation involving permutations of q-tuples involving one or more divisors are 1.

If $a^q = 0$ has a nontrivial solution, then the function g such that $g(1) = 1_k$, $g(n) = a$ for all $n > 1$ is an element of order q.

Reorderings resulting in cancelation

We can enumerate the reorderings of a q-tuple by associating each one with a partitioning of a set of q elements. In a commutative ring, each reordering gives the same product when the divisors are input. We can enumerate the reorderings of q-tuples containing 1 which give the same term in the sum by associating each one with a unique subset of a set of q elements.

Number Theory results

The congruence $ax \equiv 1 (mod\ m)$ has a solution if and only if $gcd(a, m) = 1$. If this congruence has a solution, then the integer analogue a_k is a unit, and permits cancelation.

If $gcd(a, m) \neq 1$, then the integer analogue a_k is a divisor of zero. Further research may involve determining whether or not this lack of cancelation implies the existence of nontrivial torsion elements.

Torsion in A_k for a commutative k

If $A_k(1)$ is torsion free for a commutative ring k, then we know that the only torsion elements of A_k are the trivial torsion elements of the form

$$f(n) = \begin{cases} a & n = 1 \\ 0 & \text{else} \end{cases}$$

where a is a unit of finite multiplicative order in $U(k)$.

The group $U(k)$ is isomorphic to the subgroup $F_0 \leq A_k$, where $F_0 = \{f \in A_k \mid f(n) = 0 \; \forall n > 0\}$. Consequently, the torsion group of $U(k)$ is isomorphic to the torsion subgroup of F_0, which is the torsion group of A_k.

Torsion in A_k for a noncommutative k

We get the following example of nontrivial torsion elements of order 2 in the set of all 2×2 matrices with entries in Z or Z_n. Let

$$f(n) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & n = 1 \\ \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & n \text{ is prime} \\ \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & n \text{ is composite} \end{cases}$$